Mathematical Model for Gyroscope’s Gimbal Motions

Ryspek Usubamatov

Abstract - Gyroscopes have a wide application in navigation and control systems of engineering. The main property of the gyroscope is maintaining the axis of a spinning rotor. This property is described by the mathematical model based on the law of kinetic energy conservation and changes in angular momentum. However, known mathematical models are intricate and do not match the practice of gyroscopes, but the nature of gyroscope effects is more complex. Recent investigations demonstrate that on a gyroscope are acting simultaneously and interdependently centrifugal, common inertial and Coriolis forces as well as changes in angular momentum. Centrifugal and Coriolis forces generate a resistance torque and common inertial forces and changes in angular momentum produce precession torque in gyroscopes. This paper represents a new mathematical model of acting forces. Analytical models well matched with results of practical tests.

Keywords - Gyroscopic theory, force, property, torque

I. INTRODUCTION

MORE than 200 years the gyroscope theory attracts researchers that are trying to find true solution. Gyroscopic properties are demonstrated by the spinning rotor that enables to function of numerous gyroscopic devices in engineering [1], [2]. Numerous researchers investigated, developed and added new interpretations for gyroscopic effects, which are display in the rotor’s persistence of maintaining its plane of rotation [3], [4]. All publications in the area of gyroscopic theory describe gyroscopic effects in terms of the conservation of kinetic energy and changes in angular momentum [5]. However, known mathematical models for the gyroscopic effects do not match practice of gyroscopic devices [6]-[8]. This is the reason that gyroscopic theory still attracts many researchers who seek to discover new properties for these devices [9].

The nature of gyroscopic effects is more complex. Recent investigations of the physical principles of gyroscopic motions demonstrate that centrifugal, common inertial and Coriolis forces as well as changes in the angular momentum of spinning rotors are the basis for all gyroscopic effects and properties [10], [11]. All forces and torques are interrelated and occur simultaneously. New mathematical models for acting torques accurately describe gyroscopic effects and are validated by tests. This paper represents the mathematical model for gyroscopic’s gimbal motions that represent specific unsolved property. New model validated by practical tests that conducted on the Super Precision Gyroscope model “Brightfusion Ltd”.

II. METHODOLOGY

Recent investigations in an area of the gyroscopic theory have demonstrated that, the external load torque acting on a gyroscope generates four internal torques based on action of centrifugal, common inertial and Coriolis forces as well as changes in angular momentum (Table 1). The main property of the gyroscope is maintaining the axis of a spinning rotor. This property is described by the known mathematical model based on the law of kinetic energy conservation and changes in angular momentum, which do not match the practice of gyroscopes. This paper represents a new mathematical model for physical principle of gyroscopic’s gimbal motions that was as unsolved problem. Analytical models well matched with results of practical tests.

<table>
<thead>
<tr>
<th>Type of the internal torque generated by</th>
<th>Equation, (N.m)</th>
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</thead>
<tbody>
<tr>
<td>Centrifugal forces</td>
<td></td>
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<tr>
<td>Inertial forces</td>
<td>( T_{cr} = T_{in} = \frac{2}{3} \pi J \omega \omega_{pi} )</td>
</tr>
<tr>
<td>Coriolis forces</td>
<td>( T_{cr} = (8/9) J \omega \omega_{pi} )</td>
</tr>
<tr>
<td>Change in angular momentum</td>
<td>( T_{am} = J \omega \omega_{pi} )</td>
</tr>
<tr>
<td>Resistance torque</td>
<td>( T_{r} = \left[ \frac{2}{3} \pi^2 + \frac{8}{9} \right] J \omega \omega_{pi} )</td>
</tr>
<tr>
<td>Precession torque</td>
<td>( T_{p} = T_{am} + T_{am} )</td>
</tr>
</tbody>
</table>

Symbols of the equations in Table 1 are as follows: \( T_{cr}, T_{in}, T_{cr}, \) and \( T_{am} \) are the torques generated by the centrifugal, inertial and Coriolis forces and by the change in the angular momentum respectively; \( J = (MR^2/2) \) is the rotor’s mass moment of inertia; \( M \) is the rotor’s mass; \( R \) is the external radius of the rotor, and \( \omega_{pi} \) is the angular velocity of precession around axis \( i \).

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The interrelated action of the external and the internal torques (Table 1) is represented graphically in Fig. 1. The action of the load torque generates eight internal torques acting around two axes. The sequence actions of the torques are considered step by step to get the clear picture of the gyroscope motions.

- Equations of resistance and precession torques \( (T_{ct.x} \text{ and } T_{in.x}) \) are acting around axes \( ox \) and \( oy \) simultaneously and have one angular velocity of precession \( \omega_{xy} \). The action of these torques cannot be separated. Blocking the turn of the gyroscope at any axis means the angular velocity of precession is ended. In this case, \( \omega_{xy} = 0 \) or \( \omega_{xy} = 0 \), i.e., the resistance and precession torques are deactivated. This fact is expressed as \( T_{ct.x} = T_{in.x} = 0 \).
- If the load torque is activated \( (T = 0) \), then all resistance and precession torques also are deactivated, in spite of their action on different axes.
- If the load torque \( T \) is activated and the gyroscope turns around axis \( ox \), but the turn of the gyroscope around axis \( oy \) is blocked, the action of torques is as follows:
  - The precession torques \( (T_{in.x} \text{ and } T_{am.y}) \) are activated with the angular velocity of their free turn around axis \( oy \);
  - The gyroscope’s free turn around axis \( ox \) generates the action of the centrifugal and Coriolis forces, but at the same time, they are deactivated due to the blocked turn of the gyroscope around axis \( oy \).
  - The resistance torques \( (T_{in.y} \text{ and } T_{am.x}) \) acting around axis \( ox \) that generated by the inertial forces and the change in the angular momentum acting around axis \( oy \) are deactivated;
  - The gyroscopic’s internal torques represent the internal energy that is constant for the given gyroscope parameters. New mathematical models enable to describe of all gyrooscope properties.

### III. CASE STUDY

The external torque applied to a gyroscope generates two turn motions of the spinning rotor around axis \( ox \) and \( oy \). The action of the resistance and precession torques has a defined link that should be represented by the mathematical model. Practice of gyroscop’s gimbal turns demonstrates that the angular velocities of the gimbals around the considered axes are different. The ratio of the angular velocities of the gimbals is defined by analysis of the gyroscope’s internal energy. All internal torques are represented the internal energy of the spinning rotor generated by the external torque. Internal energy of one axis is equal to the internal energy of another one. This statement for gyroscope systems (Fig. 1) is proven by the following components:

- The torque generated by the centrifugal forces acting around axis \( ox \) is equal to the torque generated by the inertial forces acting around axis \( oy \) and vise-versa.
- The torque generated by the change in the angular momentum acting around one axis is the same that acting around other one.
- The torque generated by the Coriolis forces acting around one axis is the same that acting around other one.

The sum of torques originated by axis \( ox \) is equal the sum of torques originated by axis \( oy \) is expressed by the following equation:

\[
T_{ct.x} + T_{cr.x} + T_{in.x} + T_{am.x} = T_{ct.y} + T_{cr.y} + T_{in.y} + T_{am.y} \quad (1)
\]

Substituting expressions of the torques (Table 1) into Eq. (1) and transforming yields the following equation:

\[
\begin{align*}
2 \left( \frac{\pi}{3} \right)^2 + \frac{8}{9} J \omega_{xy}^2 &= \frac{2 \left( \frac{\pi}{3} \right)^2 + 1}{J \omega_{xy}^2} \left( \frac{\pi}{3} \right)^2 + \frac{8}{9} J \omega_{xy}^2 \\
&= 2 \left( \frac{\pi}{3} \right)^2 + 1 + \frac{2 \left( \frac{\pi}{3} \right)^2 + 8}{9} J \omega_{xy}^2
\end{align*}
\]

or \( \omega_{xy} = \omega_{xy} \), i.e., angular velocities around axes are equal.

The result (2) demonstrates the equality of the angular velocities around two axes, i.e., the equality of energy of the internal torques. Each axis contains resistance and precession torques only. Replacements of torques of (1) lead to change the magnitudes of the angular velocities around axes, but energies remain constant.

The equation of internal torques acting on the gyroscope around axis \( ox \) and \( oy \) (Fig. 1) enables expressing the angular velocities of the motions of the gimbals by the following equations:

\[
\begin{align*}
-(T_{ct.x} + T_{cr.x}) - (T_{in.y} + T_{am.y}) &= 0 \\
(T_{in.x} + T_{am.x}) - (T_{ct.y} + T_{cr.y}) &= 0
\end{align*}
\]

where all parameters are as specified in Table 1.

Two (3) represent internal torques and internal energy acting on the gyroscope around axis \( ox \) and \( oy \) that is equal. This statement is formulated by substituting expressions of the torques (Table 1) into (3). Following transformation yields the following equations:
The angular dependency of the gimbals around axis oy velocities of the gimbals’ precessions around axis ox and oy is represented by the following expression:

\[ \omega_y = -\left(4\pi^2 + 17\right)\omega_x \]  

(5)

where the sign (-) means the direction of the action that can be omitted for the following considerations, and other parameters are as specified above.

Equation (5) represents the ratio between the angular velocities of the gimbals’ precessions around axis ox and oy. The angular dependency of the gimbals around axis ox and oy is represented by the following expression:

\[ \varphi = -(4\pi^2 + 17)\gamma \]  

(6)

Practical tests demonstrate the turn of the one gimbal around axis ox on the limited angle \(\gamma_{lim}\) leads to the turn of the other gimbal at \(\varphi = 90^0\) around axis oy. Substituting the defined parameters into (6) and transforming yield the magnitude of the limited angle \(\gamma_{lim}\) of the gimbal’s turn:

\[ \gamma_{lim} = \frac{90^0}{4\pi^2 + 17} = 1^035'34.8'' \]  

(8)

Practically, the magnitude of the gimbal’s turn on the angle \(\gamma_{lim} = 1^035'34.8''\) is validated by the tests of the Supper Precision Gyroscope “Brightfusion LTD” with the angular velocity of the rotor at 10000 rpm. Fig. 2 depicts the location of the spin axis and the gimbal at different stages of the tests. At the starting condition, the spin axis is located horizontally, and the gimbal with the rotor’s disc is located vertically (Fig. (2a)). Minor turn of the spinning axis around the vertical axis of the platform with three legs leads to intensive turn of the gimbal on defined angle (Fig. (2b)). The following turn of the spinning axis on angle \(\gamma_{lim}\) leads to the turn of the gimbal that finally comes to a horizontal location (Fig. 2(c)). The turn of the spinning axis more than \(\gamma_{lim}\) does not lead to a change in the horizontal location of the gimbal (Fig. (2d)).

Mathematical models for gyroscope internal torques and gimbals’ motions around axis ox and oy describe the known property that validated by practical test.

IV. RESULTS AND DISCUSSION

External torque applied to a gyroscope leads to an angular velocity of precessions and generates internal torques based on actions of the centrifugal, common inertial and Coriolis forces as well as the change in the angular momentum of the spinning rotor. New torques are represented the gyroscope’s internal energy that distributed equally by two axes. This internal energy generates the constant ratio of the gyroscope’s turns around two axes. Physical principle of acting internal torques is described by mathematical model that enables to compute the gimbals motions around axes. Practical test validates the analytical approach.
V. CONCLUSION

Gyroscopic theory in classical mechanics is one of the most complex and intricate in terms of analytical solutions. The known mathematical models for gyroscope theory are mainly based on the actions of the external torque applied and the change in the angular momentum of the spinning rotor. This approach leads to assumptions and simplifications for solving unexplainable motions of gyroscopic devices. Practically, gyroscope effects demonstrate that the torques generated by the centrifugal, common inertial and Coriolis forces and as well as the change in the angular momentum of the spinning rotor play a critical role. The action of these forces represent the internal energy of a gyroscope that formulated by mathematical model. The presentation of the internal torques enables clearly understanding the physical process that results in motions of gyroscopic devices. New mathematical models of the gyroscope effects clearly describe the known properties.

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REFERENCES