Relations between the Pseudo-Integral and Some Pseudo-Type Integral Transforms Based on Special Pseudo-Operations

Dhurata Valera

***Abstract*—**The generalization of the pseudo-integration type transform and the pseudo-exhange formula are proved in the specials cases of the semirings (G,⊕,⊙), based on the special generated (⊕,⊙,⊗)-operators. Many results give the properties of the pseudo-integration type transform and inverse of the pseudo-integration type transform (Pseudo-Laplase Transform, Pseudo-Fourier Transform), also the relations with the pseudo-integral and the classical transform. The results can be applied in dynamical programming and some differential equations.

***Keywords*—**Pseudo-integral transform, semiring, (⊕, ⊙, ⊗)-operators, Pseudo-Laplase Transform, Pseudo-Fourier Transform.

I. Preliminary Notions

The binary operations (⊕,⊙) (pseudo-addition, pseudo-multiplication) are respectively [1], [2], [7], [9] the functions

$\oplus: [0,+\infty] \times [0,+\infty] \to [0,+\infty]$ and

$\odot: [0,+\infty] \times [0,+\infty] \to [0,+\infty]$

with following axioms that fulfill [8], [15], [16], [18], [20], [25], [26], [27], [28]:

⊕(A.1÷A.8) (*Commutative; Associative; Monotonitive; Continuitive; With a neutral element* denote $0_{\oplus}$*; Arkimedian property; Finiteness axiom; Properties respect to ordinary operations* (+, ·)(Or. A., Or. M.)).

⊙(A.1÷A.8) (*Right distributive over* ⊕*; Positively non-decreasing*; *Pseudo-multiplication with 0;* There exist a *left unit e*, (denote e $=1_{\odot}$); *Continuity; Commutative; Associative; Left distributive over* ⊕).

Let a generator $g: [-\infty,+\infty] \longrightarrow [-\infty,+\infty]$ be a (***CSI***) continuous, strictly increaing function of the pseudo-addition ⊕ on interval $[-\infty,+\infty]$ such that $g(0) = 0_{\oplus}$, $g(1) = 1_{\odot}$, $g(+\infty) = +\infty$ or an odd extension of a given generator g from $[0, +\infty]$ to $[-\infty,+\infty]$).

The operations of pseudo-substraction and pseudo- division were introduced by Mesiar and Rybárik [8], [23], [25].

Dhurata Valera is with the "Aleksandër Xhuvani" University, NSF, Mathematics Department, Elbasan, Albania. (corresponding author's e-mail: dhurata_valera@hotmail.com).

Definition 1.1 Let a function g be a *generator* of a pseudo-addition ⊕ on the interval $[-\infty,+\infty]$. Binary operation ⊖ and ⊘ on $[-\infty,+\infty]$ defined by the formulas:

$x \ominus y = g^{-1}\left(g(x) - g(y)\right)$, $x \oslash y = g^{-1}\left(g(x)/ g(y)\right)$,

(if the expressions $\left(g(x) - g(y)\right)$ and $\left(g(x)/ g(y)\right)$ have sense) is said to be *pseudo-substraction* and *pseudo-division* consistent with the pseudo-addition ⊕ [2], [5], [6], [7], [8], [13], [14], [15], [17], [19].

Than the sistem of pseudo-arithmetical operations {⊕,⊙,⊖,⊘} generated by this function g is said to be a *consistent* sistem [8].

So for $x, y \in [-\infty,+\infty]$ and let g be a generator on $[-\infty,+\infty]$ we put [36]:

$\oplus(x,y) = g^{-1}\left((g \circ \oplus)(x,y)\right) = g^{-1}\left(g(x) + g(y)\right)$

$\odot(x,y) = g^{-1}\left((g \circ \odot)(x,y)\right) = g^{-1}\left(g(x) \cdot g(y)\right)$

$\ominus(x,y) = g^{-1}\left((g \circ \ominus)(x,y)\right) = g^{-1}\left(g(x) - g(y)\right)$

$\oslash(x,y) = g^{-1}\left((g \circ \oslash)(x,y)\right) = g^{-1}\left(g(x)/ g(y)\right)$

(With some valued undefined [36]).

The structure (G ⊕, ⊙) is called a *semiring* ($G \subseteq [-\infty,+\infty]$; $G = G_1, G = G_{2g}$ *or* $G = G_3$)) [1], [2], [3], [5], [9], [10], [15], [21], [22], [24], [27], [29].

We will consider the very special semirings with: $G \in \{[0,+\infty], [0,1], [1,+\infty], [a,b], [-\infty,+\infty]\}$ and the continuous pseudo-operations (⊕,⊙) [4], [9], [12] , [15].

Class 1. $(SMR_1 - G_1^{(\oplus,\odot)})$, $(G_1, \oplus, \odot)$,

$\oplus - \boldsymbol{ID}$, idempotent, $\odot - \boldsymbol{NID}$, nonidempotent

$\oplus \in \{\max, \sup; \min, \inf\}$,

$\odot = (+)$ Or. A. (ordinary addition).

Class 2. $(SMR_g - G_{2g}^{(\oplus,\odot)})$, $(G_{2g}, \oplus, \odot)$,

$\oplus - \boldsymbol{CSI}$, strict;

$\odot - \boldsymbol{CSI}$, strict.

$\boldsymbol{g} - \mathbf{CSI}$, Continuous and strictly increasing generator,

Class 3. $(SMR_3 - G_3^{(\oplus,\odot)})$, $(G_3, \oplus, \odot)$,

$\oplus - \boldsymbol{ID}$, idempotent; $\oplus \in \{\max, \min\}$,

$\odot - \boldsymbol{ID}$, idempotent, $\odot \in \{\max, \min\}$.

II. (⊕, ⊙)-Transform On A Semiring

- (⊕, ⊙)- ***Fourier transform***

Let $(SMR - G^{(\oplus,\odot)})$, $(SMR_1 - G_1{}^{(\oplus,\odot)})$, $(G\oplus,\odot)$ be $G=[0,1]$ a semiring, where the pseudo-operators $(\oplus,\odot)$ are given by generating function $g:[0,1]\to[-\infty,+\infty]$. Any function $f:\mathbb{R}\to[0,1]$ can be split into two parts with respect to $\oplus$ [4], [10], [11], [14], [29]:

$$f_{\oplus}(x)=f(x)=\left(g^{-1}(2^{-1})\odot(f(x)\oplus f(-x))\right)\oplus \oplus\left(g^{-1}(2^{-1})\odot(f(x)\ominus f(-x))\right)$$

$$f_{\oplus}(x)=f(x)=\oplus\left(E_{(\oplus,\odot)}(x),O_{(\oplus,\odot)}(x)\right)= =E_{(\oplus,\odot)}(x)\oplus O_{(\oplus,\odot)}(x)$$

$$E_{(\oplus,\odot)}(x)=\odot\left(g^{-1}(2^{-1}),(f(x)\oplus f(-x))\right)= =g^{-1}(2^{-1})\odot(f(x)\oplus f(-x))$$

and

$$O_{(\oplus,\odot)}(x)=\odot\left(g^{-1}(2^{-1}),(f(x)\oplus f(-x))\right)= =g^{-1}(2^{-1})\odot\big(f(x)\ominus f(-x)\big).$$

Note: The functions $E_{(\oplus,\odot)}$ and $g\circ E_{(\oplus,\odot)}$ are even functions and the function $O_{(\oplus,\odot)}$ is an even function.

Definition 2.1. The Pseudo-Fourier cosine transform $\mathcal{F}_c^{(\oplus,\odot)}$ on the semiring $(SMR-[0,1]^{(\oplus,\odot)})$, of a real *measurable function* $f:\mathbb{R}\to[0,1]$ is:

$$\mathcal{F}_c^{(\oplus,\odot)}[f(x)](\alpha)= =g^{-1}\left(\frac{1}{\sqrt{2\pi}}\right)\odot\int_{[-\infty,+\infty]}^{(\oplus,\odot)}g^{-1}\left(\cos(\alpha x)\odot E_{(\oplus,\odot)}(x)dx\right.$$

for every number α ([4], [9], [11] if the right side exist).

$$\mathcal{F}_c^{(\oplus,\odot)}[f(x)](\alpha)= =\odot\left(g^{-1}\left(\tfrac{1}{\sqrt{2\pi}}\right),\int_{[-\infty,+\infty]}^{(\oplus,\odot)}g^{-1}\left(\cos(\alpha x)\odot E_{(\oplus,\odot)}(x)dx\right)\right.$$

Definition 2.2. The Pseudo-Fourier sine transform $\mathcal{F}_c^{(\oplus,\odot)}$ on the semiring $(SMR-[0,1]^{(\oplus,\odot)})$, of a *real measurable function* $f:\mathbb{R}\to[0,1]$ is:

$$\mathcal{F}_c^{(\oplus,\odot)}[f(x)](\alpha)= =g^{-1}\left(\tfrac{1}{\sqrt{2\pi}}\right)\odot\int_{[-\infty,+\infty]}^{(\oplus,\odot)}g^{-1}\left(\sin(\alpha x)\odot O_{(\oplus,\odot)}(x)dx\right.,$$

for every real number α (if the right side exist).

$$\mathcal{F}_c^{(\oplus,\odot)}[f(x)](\alpha)= =\odot\left(g^{-1}\left(\frac{1}{\sqrt{2\pi}}\right),\int_{[-\infty,+\infty]}^{(\oplus,\odot)}g^{-1}\left(\sin(\alpha x)\odot O_{(\oplus,\odot)}(x)dx\right)\right.$$

Integrals on the transforms are $(\oplus,\odot)$-integrals based on the $(SMR_g-G_{2g}{}^{(\oplus,\odot)})=(SMR_g-[0,1]_{2g}{}^{(\oplus,\odot)})$, so the pseudo-Fourier cosine$(\boldsymbol{Furie},g-\mathcal{F}_c^{(\oplus,\odot)})=\mathcal{F}_{cg}^{(\oplus,\odot)}$ and sine transform $(\boldsymbol{Furie},g-\mathcal{F}_s^{(\oplus,\odot)})=\mathcal{F}_{sg}^{(\oplus,\odot)}$.

By the g −integrals pseudo-Fourier transform give us the following forms for two types:

$$(\boldsymbol{Furie},g-\mathcal{F}_c^{(\oplus,\odot)})=\mathcal{F}_{cg}^{(\oplus,\odot)}[f(x)](\alpha)=$$

$$=g^{-1}\left(\tfrac{1}{\sqrt{2\pi}}\right)\odot\int_{[-\infty,+\infty]}^{(\oplus,\odot)}g^{-1}\left(E_{(\oplus,\odot)}(x)\right)\cos(\alpha x)dx.$$

$$(\boldsymbol{Furie},g-\mathcal{F}_s^{(\oplus,\odot)})=\mathcal{F}_{sg}^{(\oplus,\odot)}[f(x)](\alpha)= =g^{-1}\left(\frac{1}{\sqrt{2\pi}}\right)\odot\int_{[-\infty,+\infty]}^{(\oplus,\odot)}g^{-1}\left(O_{(\oplus,\odot)}(x)\right)\sin(\alpha x)dx.$$

The pseudo-Frourier transform $\mathcal{F}^{(\oplus,\odot)}$ of some real measurable function is expressed in terms of $\mathcal{F}_c^{(\oplus,\odot)}$ and $\mathcal{F}_s^{(\oplus,\odot)}$, as in the classical case [4], [9], [11], [14]. **Definition** 2.3 The Pseudo-Fourier transform $\mathcal{F}^{(\oplus,\odot)}$ based on the semiring $(SMR_g-G_{2g}{}^{(\oplus,\odot)})=(SMR_g-[0,1]_{2g}{}^{(\oplus,\odot)})$, of a *real measurable function* $f:\mathbb{R}\to[0,1]$, for every real number α, is:

$$(\boldsymbol{Furie},g-\mathcal{F}^{(\oplus,\odot)})(f-RMF)= =(\boldsymbol{Furie},g-\mathcal{F}_c^{(\oplus,\odot)})(f-RMF)- -i(\boldsymbol{Furie},g-\mathcal{F}_s^{(\oplus,\odot)})(f-RMF),$$

or briefly

$$\mathcal{F}_g^{(\oplus,\odot)}[f(x)](\alpha)=\mathcal{F}_{cg}^{(\oplus,\odot)}[f(x)](\alpha)-i\mathcal{F}_{sg}^{(\oplus,\odot)}[f(x)](\alpha).$$

- The ***basic properties*** of this $(\boldsymbol{Furie},g-\mathcal{F}^{(\oplus,\odot)})$ – type transform

1. For *real measurement function* $f,h:\mathbb{R}\to[0,1]$, $(f,h-RMF)$ and a and b some real parameters, then for $(\boldsymbol{Furie},g-\mathcal{F}^{(\oplus,\odot)})$ – type transform we obtain [4], [9], [11] the *pseudo-linearity property*:

$$\mathcal{F}_g^{(\oplus,\odot)}[\odot(a,f)]=\odot\left((a),(\mathcal{F}_g^{(\oplus,\odot)}[f])\right)= =\odot\left((a),(\mathcal{F}_{cg}^{(\oplus,\odot)}[f])\right)-i\left(\odot\left((a),(\mathcal{F}_{sg}^{(\oplus,\odot)}[f])\right)\right)$$

$$\mathcal{F}_g^{(\oplus,\odot)}[\oplus(f,h)]= =\oplus\left((\mathcal{F}_{cg}^{(\oplus,\odot)}[f]),(\mathcal{F}_{cg}^{(\oplus,\odot)}[h])\right)- -i\left(\oplus\left((\mathcal{F}_{sg}^{(\oplus,\odot)}[f]),(\mathcal{F}_{sg}^{(\oplus,\odot)}[h])\right)\right)$$

$$\mathcal{F}_g^{(\oplus,\odot)}[f(x)\oplus h(x)](\alpha)= =\left(\left(\mathcal{F}_{cg}^{(\oplus,\odot)}[f(x)](\alpha)\right)\oplus\left(\mathcal{F}_{cg}^{(\oplus,\odot)}[h(x)](\alpha)\right)\right)- -i\left(\left(\mathcal{F}_{sg}^{(\oplus,\odot)}[f(x)](\alpha)\right)\oplus\left(\mathcal{F}_{sg}^{(\oplus,\odot)}[h(x)](\alpha)\right)\right).$$

2. For real measurement function $f:\mathbb{R}\to[0,1]$, $(f-RMBF)$ and b some real parameters [4], [9], [11], [14] then for

$(\boldsymbol{Furie}, g - \mathcal{F}^{(\oplus,\odot)})$ – type transform we obtain the *pseudo-shift property*:

$$(\boldsymbol{Furie}, g - \mathcal{F}_c^{(\oplus,\odot)})(f - RMBF) = \\ = \mathcal{F}_{cg}^{(\oplus,\odot)}[f(x-b)](\alpha) = \\ = \left(g^{-1}(\cos b\alpha) \odot \mathcal{F}_{cg}^{(\oplus,\odot)}[f(x)](\alpha)\right) \ominus \\ \ominus \left(g^{-1}(\sin b\alpha) \odot \mathcal{F}_{sg}^{(\oplus,\odot)}[f(x)](\alpha)\right)$$

$$(\boldsymbol{Furie}, g - \mathcal{F}_s^{(\oplus,\odot)})(f - RMBF)) = \\ = \mathcal{F}_{sg}^{(\oplus,\odot)}[f(x-b)](\alpha) = \\ = \left(g^{-1}(\cos b\alpha) \odot \mathcal{F}_{sg}^{(\oplus,\odot)}[f(x)](\alpha)\right) \ominus \\ \ominus \left(g^{-1}(\sin b\alpha) \odot \mathcal{F}_{cg}^{(\oplus,\odot)}[f(x)](\alpha)\right).$$

3. For *real measurement function* $f : \mathbb{R} \to [0,1]$, $(f - RMBF)$ ($f -$ real mearurable and bounded is in sence that $\lim_{x\to+\infty} f(x) = 0$ with respect to some pseudo-metric based on *generator* g) , then for $(\boldsymbol{Furie}, g - \mathcal{F}^{(\oplus,\odot)})$ – type transform we obtain the *pseudo-derivative and pseudo-convolution*:

$$\mathcal{F}_{cg}^{(\oplus,\odot)}\left[\frac{d^{\oplus} f(x)}{dx}\right](\alpha) = g^{-1}(\alpha) \odot \mathcal{F}_{cg}^{(\oplus,\odot)}[f(x)](\alpha) = \\ = \odot\left(\left(g^{-1}(\alpha)\right), \left(\mathcal{F}_{cg}^{(\oplus,\odot)}[f(x)](\alpha)\right)\right)$$

$$\mathcal{F}_{sg}^{(\oplus,\odot)}\left[\frac{d^{\oplus} f(x)}{dx}\right](\alpha) = g^{-1}(\alpha) \odot \mathcal{F}_{sg}^{(\oplus,\odot)}[f(x)](\alpha) = \\ = \odot\left(\left(g^{-1}(\alpha)\right), \left(\mathcal{F}_{sg}^{(\oplus,\odot)}[f(x)](\alpha)\right)\right)$$

where $\frac{d^{\oplus}}{dx}$ is pseudo-derivative by [2], [9], [10], [11], [13], [14].

Theorem 2.4 The Pseudo-Fourier transform $(\boldsymbol{Furie}, g - \mathcal{F}^{(\oplus,\odot)})(f, h - RMF)$ based on the semiring $(SMR_g - G_{2g}{}^{(\oplus,\odot)}) = (SMR_g - [0,1]_{2g}{}^{(\oplus,\odot)})$, of *real measurable function* $f, h : \mathbb{R} \to [0,1]$, for every real number α, is

$$(\boldsymbol{Furie}, g - \mathcal{F}_c^{(\oplus,\odot)})(f, h - RMF) = g^{-1}(\sqrt{2\pi}) \odot \\ \odot \left(\left(\mathcal{F}_{cg}^{(\oplus,\odot)}[f] \odot \mathcal{F}_{cg}^{(\oplus,\odot)}[h]\right) \ominus \left(\mathcal{F}_{sg}^{(\oplus,\odot)}[f] \odot \mathcal{F}_{sg}^{(\oplus,\odot)}[h]\right)\right)$$

$$(\boldsymbol{Furie}, g - \mathcal{F}_s^{(\oplus,\odot)})(f, h - RMF) = g^{-1}(\sqrt{2\pi}) \odot \\ \odot \left(\left(\mathcal{F}_{cg}^{(\oplus,\odot)}[f] \odot \mathcal{F}_{cg}^{(\oplus,\odot)}[h]\right) \ominus \left(\mathcal{F}_{sg}^{(\oplus,\odot)}[f] \odot \mathcal{F}_{sg}^{(\oplus,\odot)}[h]\right)\right).$$

- ***Inverse pseudo-Fourier Transform***

The inverse pseudo-Fourier Transform show that the transform from $(\boldsymbol{Furie}, g - \mathcal{F}^{(\oplus,\odot)})(f, h - RMF)$ can be back to functions by inverse transformation:

$$F(\alpha) = \mathcal{F}[g \circ f](\alpha) = \frac{1}{\sqrt{2\pi}} \int_{[-\infty,+\infty]}^{(\oplus,\odot)} g \circ f(t)\, e^{-i\alpha t}\, dt = \\ = \frac{1}{\sqrt{2\pi}} \left(\int_{[-\infty,+\infty]}^{(\oplus,\odot)} g \circ E_{(\oplus,\odot)}(t) \cos \alpha t\, dt - i\, \frac{1}{\sqrt{2\pi}} \int_{[-\infty,+\infty]}^{(\oplus,\odot)} g \circ O_{(\oplus,\odot)}(t) \sin \alpha t\, dt \right) = \\ = g \circ \mathcal{F}_{cg}^{(\oplus,\odot)}[f(x)](\alpha) - i\, g \circ \mathcal{F}_{sg}^{(\oplus,\odot)}[f(x)](\alpha)$$

where $\mathcal{F}$ is the *classical Fourier transform*.

Both $\mathcal{F}_{cg}^{(\oplus,\odot)}[f(x)](\alpha)$ and $g \circ \mathcal{F}_{cg}^{(\oplus,\odot)}[f(x)](\alpha)$ are odd.

$$\int_{[-\infty,+\infty]}^{(\oplus,\odot)} p(x) \odot dm_q(x) = \\ = g^{-1}\left(\frac{1}{\sqrt{2\pi}}\right) \odot \int_{[-\infty,+\infty]}^{(\oplus,\odot)} p(x) \odot q(x)$$

and

$$\int_{[-\infty,+\infty]}^{(\oplus,\odot)} p(x) \odot d(x) = \left(\frac{1}{\sqrt{2\pi}}\right) \odot \int_{[-\infty,+\infty]}^{(\oplus,\odot)} p(x) \odot d(x).$$

In analogy with classical analysis is obtain the *inverse transform* on following form:

$$\left(\mathcal{F}_g^{(\oplus,\odot)}\right)^{-1}\left[\mathcal{F}_g^{(\oplus,\odot)}[f]\right](x) = \\ = \int_{[-\infty,+\infty]}^{(\oplus,\odot)} g^{-1}(\cos tx) \odot dm_{\mathcal{F}_{cg}^{(\oplus,\odot)}}(t) \oplus \\ \oplus \int_{[-\infty,+\infty]}^{(\oplus,\odot)} g^{-1}(\sin tx) \odot dm_{\mathcal{F}_{sg}^{(\oplus,\odot)}}(t) = \\ = \left(\mathcal{F}_{cg}^{(\oplus,\odot)}\right)^{-1}\left[\mathcal{F}_{cg}^{(\oplus,\odot)}[f]\right](x) \oplus \left(\mathcal{F}_{sg}^{(\oplus,\odot)}\right)^{-1}\left[\mathcal{F}_{sg}^{(\oplus,\odot)}[f]\right](x) = \\ = E_{(\oplus,\odot)}(x) \oplus O_{(\oplus,\odot)}(x).$$

Lemma 2.5 If $g \circ f \in L^1$ then $\mathcal{F}_g^{(\oplus,\odot)}[f]$ is continuous.

Lemma 2.6 Let be $h_\lambda(x) = \int_{[0,\ \lambda]}^{(\oplus,\odot)} \cos(tx)\, d(t)$ and $(f \otimes g)(x) = g^{-1}\left(\int_{[-\infty,+\infty]}^{(\oplus,\odot)} g \circ f(x-y) h(y)\, d(y)\right)$.
If $g \circ f \in L^1$ then

$$(f \otimes h_\lambda)(x) = \left(\int_{[0,\ \lambda]}^{(\oplus,\odot)} g^{-1}(\cos tx) \odot dm_{\mathcal{F}_{cg}^{(\oplus,\odot)}}(t)\right) \oplus \\ \oplus \left(\int_{[0,\ \lambda]}^{(\oplus,\odot)} g^{-1}(\sin tx) \odot dm_{\mathcal{F}_{cg}^{(\oplus,\odot)}}(t)\right).$$

Theorem 2.7 If $g \circ f \in L^1$ and $g \circ \mathcal{F}_{cg}^{(\oplus,\odot)}[f(x)] \in L^1$, $g \circ \mathcal{F}_{sg}^{(\oplus,\odot)}[f(x)] \in L^1$ then

$$\lim_{\lambda\to\infty}(f \otimes h_\lambda)(x) = \left(\mathcal{F}_g^{(\oplus,\odot)}\right)^{-1}\left[\mathcal{F}_g^{(\oplus,\odot)}[f]\right](x) = f(x),$$

nearly everywhere and $\left(\mathcal{F}_g^{(\oplus,\odot)}\right)^{-1}\left[\mathcal{F}_g^{(\oplus,\odot)}[f]\right]$ is continuous.

Proof.

$$\lim_{\lambda\to\infty}(f \otimes h_\lambda)(x) = \lim_{\lambda\to\infty}\left(\left(\int_{[0,\ \lambda]}^{(\oplus,\odot)} g^{-1}(\cos\ tx) \odot dm_{\mathcal{F}_{cg}^{(\oplus,\odot)}}(t)\right) \oplus \left(\int_{[0,\ \lambda]}^{(\oplus,\odot)} g^{-1}(\sin\ tx) \odot dm_{\mathcal{F}_{cg}^{(\oplus,\odot)}}(t)\right)\right) =$$

$$= \oplus\left(\left(\left(\mathcal{F}_{cg}^{(\oplus,\odot)}\right)^{-1}\left[\mathcal{F}_{cg}^{(\oplus,\odot)}[f]\right](x)\right), \left(\left(\mathcal{F}_{sg}^{(\oplus,\odot)}\right)^{-1}\left[\mathcal{F}_{sg}^{(\oplus,\odot)}[f]\right]\right)\right) =$$

$$= \oplus\left(E_{(\oplus,\odot)}(x), O_{(\oplus,\odot)}(x)\right) =$$

$$= E_{(\oplus,\odot)}(x) \oplus O_{(\oplus,\odot)}(x) = f_\oplus(x) = f(x).$$

- $(\oplus,\ \odot)$-***Laplas transform***

Definition 2.8 Pseudo- integral transform $\mathcal{L}^{(\oplus,\odot)}$ of a *real measurable function* based on the semiring $(SMR_g - G_{2g}{}^{(\oplus,\odot)}) = (SMR_g - [0,1]_{2g}{}^{(\oplus,\odot)})$, of a *real measurable function* $f : \mathbb{R} \to [0,1]$, for every [4], [13] real number α, is

$$(\boldsymbol{Laplase}, g - \mathcal{L}^{(\oplus,\odot)})(f - RMF) = \int_G^{(\oplus,\odot)} g(x, -z) \odot dm_f(x)$$

$(\boldsymbol{Laplase}, g - \mathcal{L}^{(\oplus,\odot)})$ is pseudo-linear:

$$\begin{aligned}\mathcal{L}^{(\oplus,\odot)}[(a\odot f) \oplus (b\odot h)] &= \\ &= a\odot\mathcal{L}^{(\oplus,\odot)}[f] \oplus b\odot\mathcal{L}^{(\oplus,\odot)}[h] = \\ &= \oplus\left(\left(a\odot\mathcal{L}^{(\oplus,\odot)}[f]\right), \left(b\odot\mathcal{L}^{(\oplus,\odot)}[h]\right)\right)\end{aligned}$$

where

$$\begin{aligned}\mathcal{L}^{(\oplus,\odot)}[f \otimes h](z) &= \left(\mathcal{L}^{(\oplus,\odot)}[f](z)\right)\odot\left(\mathcal{L}^{(\oplus,\odot)}[h](z)\right) = \\ &= \odot\left(\left(\mathcal{L}^{(\oplus,\odot)}[f]\right), \left(\mathcal{L}^{(\oplus,\odot)}[h](z)\right)\right).\end{aligned}$$

$$\begin{aligned}\left(\mathcal{L}^{(\oplus,\odot)}\right)^{-1}&\left[\left(a\odot\mathcal{L}^{(\oplus,\odot)}[f](z)\right) \oplus \left(b\odot\mathcal{L}^{(\oplus,\odot)}[h](z)\right)\right] = \\ &= \oplus\left((a\odot f), (a\odot h)\right) = \\ &= \left(a\odot\left(\left(\mathcal{L}^{(\oplus,\odot)}\right)^{-1}\mathcal{L}^{(\oplus,\odot)}[f](z)\right)\right) \oplus \\ &\oplus \left(a\odot\left(\left(\mathcal{L}^{(\oplus,\odot)}\right)^{-1}\mathcal{L}^{(\oplus,\odot)}[h](z)\right)\right)\end{aligned}$$

i.e., the inverse of the *pseudo-linear transform* $\left(\mathcal{L}^{(\oplus,\odot)}\right)^{-1}$ is also *pseudo-linear*.

Definition 2.9 For the *pseudo-linear transform* $\mathcal{L}^{(\oplus,\odot)}[f]$ *we have* briefly:

$$\mathcal{L}^{(\oplus,\odot)}[f](x) = min_{x\geq 0}\left[g^{-1}\left(e^{-zx}(g\circ f)(x)\right)\right] = g^{-1}\left(min_{x\geq 0}\left[e^{-zx}(g\circ f)(x)\right]\right)$$

$$\mathcal{L}^{(\oplus,\odot)}[f](x) = max_{x\geq 0}\left[g^{-1}\left(e^{-zx}(g\circ f)(x)\right)\right] = g^{-1}\left(max_{x\geq 0}\left[e^{-zx}(g\circ f)(x)\right]\right)$$

$$\mathcal{L}^{(\oplus,\odot)}[f](x) = min_{x\geq 1}\left[g^{-1}\left(x^{-z}(g\circ f)(x)\right)\right] = g^{-1}\left(min_{x\geq 1}\left[x^{-z}(g\circ f)(x)\right]\right)$$

$$\mathcal{L}^{(\oplus,\odot)}[f](x) = max_{x\geq 1}\left[g^{-1}\left(x^{-z}(g\circ f)(x)\right)\right] = g^{-1}\left(max_{x\geq 1}\left[x^{-z}(g\circ f)(x)\right]\right).$$

Theorema 2.10 The inverse $\left(\mathcal{L}^{(\oplus,\odot)}\right)^{-1}$ of the *pseudo-linear transform* is [2], [4], [9], [12],[14], [29] on the form**:**

$$\begin{aligned}\left(\left(\mathcal{L}^{(\oplus,\odot)}\right)^{-1}\left[\mathcal{L}^{(\oplus,\odot)}[f]\right](z)\right)(x) &= \\ &= max_{x\geq 0}\left[g^{-1}\left(e^{zx}\left(g\circ\left(\mathcal{L}^{(\oplus,\odot)}[f](z)\right)\right)(x)\right)\right] = \\ &= g^{-1}\left(max_{x\geq 0}\left[e^{zx}\left(g\circ\left(\mathcal{L}^{(\oplus,\odot)}[f](z)\right)\right)(x)\right]\right)\end{aligned}$$

$$\begin{aligned}\left(\left(\mathcal{L}^{(\oplus,\odot)}\right)^{-1}\left[\mathcal{L}^{(\oplus,\odot)}[f]\right](z)\right)(x) &= \\ &= min_{x\geq 0}\left[g^{-1}\left(e^{zx}\left(g\circ\left(\mathcal{L}^{(\oplus,\odot)}[f](z)\right)\right)(x)\right)\right] = \\ &= g^{-1}\left(min_{x\geq 0}\left[e^{zx}\left(g\circ\left(\mathcal{L}^{(\oplus,\odot)}[f](z)\right)\right)(x)\right]\right)\end{aligned}$$

$$\begin{aligned}\left(\left(\mathcal{L}^{(\oplus,\odot)}\right)^{-1}\left[\mathcal{L}^{(\oplus,\odot)}[f]\right](z)\right)(x) &= \\ &= max_{x\geq 1}\left[g^{-1}\left(x^{z}\left(g\circ\left(\mathcal{L}^{(\oplus,\odot)}[f](z)\right)\right)(x)\right)\right] = \\ &= g^{-1}\left(max_{x\geq 1}\left[x^{z}\left(g\circ\left(\mathcal{L}^{(\oplus,\odot)}[f](z)\right)\right)(x)\right]\right)\end{aligned}$$

$$\begin{aligned}\left(\left(\mathcal{L}^{(\oplus,\odot)}\right)^{-1}\left[\mathcal{L}^{(\oplus,\odot)}[f]\right](z)\right)(x) &= \\ &= min_{x\geq 1}\left[g^{-1}\left(x^{z}\left(g\circ\left(\mathcal{L}^{(\oplus,\odot)}[f](z)\right)\right)(x)\right)\right] = \\ &= g^{-1}\left(min_{x\geq 1}\left[x^{z}\left(g\circ\left(\mathcal{L}^{(\oplus,\odot)}[f](z)\right)\right)(x)\right]\right)\end{aligned}$$

Theorem 2.11 If $\mathcal{L}^{(\oplus,\odot)}[f]$ is a pseudo-Laplase transform on $(SMR - G^{(\oplus,\odot)}) = (SMR - [0,+\infty]^{(max,+)})$ then exist the inverse of pseudo-Laplase transform in the following form [1], [2], [4], [9], [10], [12],[14], [29]

$$f(x) = \left(\left(\mathcal{L}^{(\oplus,\odot)}\right)^{-1}\left[\mathcal{L}^{(\oplus,\odot)}[f]\right](z)\right)(x) = inf_{z\geq 0}\left[xz + \left(\mathcal{L}^{(\oplus,\odot)}[f](z)\right)(x)\right]$$

$$f(x) = max_G(f_1 \otimes f_2 \otimes \ldots \otimes f_n)(x) = max_G\{f_1(x_1) \odot f_2(x_2) \odot \ldots \odot f_n(x_n)\} \overset{\odot=+}{\Longleftrightarrow} \overset{\odot=+}{\Longleftrightarrow} max_G\left(\sum_{i=1}^n f_i(x_i)\right)$$

$$f(x) = \left(\left(\mathcal{L}^{(\oplus,\odot)}\right)^{-1}\left[\sum_{i=1}^n \mathcal{L}^{(\oplus,\odot)}[f_i](x_i)\right](z)\right)(x) = min_{z\geq 0}\left[xz + \left(\sum_{i=1}^n \mathcal{L}^{(\oplus,\odot)}[f](z)\right)(x)\right].$$

III. Applying the Inverse of the Pseudo-Integral Transform

Let be $U(x_1, x_2, \ldots, x_n) = f_1(x_1) \odot f_2(x_2) \odot \ldots \odot f_n(x_n) = \odot_{i=1}^n f_i(x_i)$ and

$f(x) = (f_1 \otimes f_2 \otimes \ldots \otimes f_n)(x) = \otimes_{i=1}^n (f_i)(x)$

where $\otimes$ is the pseudo-convolution [1], [2], [4],[9], [10], [12], [14], [23], [29].

We have by the *pseudo-transform* the formula:

$$\mathcal{L}^{(\oplus,\odot)}[f](z) = \mathcal{L}^{(\oplus,\odot)}[(f_1 \otimes f_2 \otimes \ldots \otimes f_n)](z) =$$

$$= \odot_{i=1}^{n} \mathcal{L}^{(\oplus,\odot)}[f_i](z).$$

Applying the inverse of the pseudo-integral transform, we obtain the *formal solution*:

$$f(x) = \left(\left(\left(\mathcal{L}^{(\oplus,\odot)}\right)^{-1}\left[\mathcal{L}^{(\oplus,\odot)}[f]\right](z)\right)(x)\right) =$$

$$= \left(\left(\mathcal{L}^{(\oplus,\odot)}\right)^{-1}\left[\odot_{i=1}^{n} \mathcal{L}^{(\oplus,\odot)}[f_i](z)\right]\right)(x) =$$

$$= \left(\left(\mathcal{L}^{(max,\odot)}\right)^{-1}\left[\odot_{i=1}^{n} \mathcal{L}^{(\oplus,\odot)}[f_i](z)\right]\right)(x).$$

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Dh.Valera is with the Mathematics Department, "Aleksandër Xhuvani" University, Elbasan, Albania and Members of EMS, AMS, AWM, ACM. (phone: 00355-54252771; e-mail: dhurata_valera@hotmail.com).

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