Backstepping Algorithm with Sliding Mode Control for Input-Affine Nonlinear Systems

Nitesh Meena, and Bharat Bhushan Sharma

***Abstract*----**This paper presents the sliding mode control methodology with backstepping algorithm to derive the structure of controller for input-affine nonlinear systems. The proposed controller guarantees the asymptotic regulation of the states of the systems to their desired values. The control scheme has been implemented for two examples of input-affine nonlinear dynamical systems i.e. a single link manipulator system and a ball and beam setup coupled with servomotor. The simulation results are presented to justify the effectiveness of control scheme and asymptotic regulation of system states to their desired trajectories.

***Keywords*---**Backstepping design, Ball and beam setup, Single link manipulator, Sliding mode control.

I. INTRODUCTION

SLIDING mode control [1-6] has attracted interest of research community because of effectiveness of fast calculation and switching actions realized through the progress of microelectronics and power electronics devices. These advancements have led to remarkable progress in designing sliding mode controllers while designing suitable switching surfaces. The application of sliding mode control to practical systems such as robotic control [7] and motor control [8] have resulted in many theoretical studies in last few decades. These studies have shown that a sliding mode controller provides better transient response than classical controllers. Sliding mode control has a distinctive feature that it can deal with nonlinearities of control systems and offers robust control design. However, since it has several important drawbacks, particularly high control gain effect due to nonlinear compensation and control chattering, an effective sliding mode controller design for nonlinear system is highly desired. The sliding mode control action law at any moment is determined by a switching condition to force the system to evolve on the sliding surface so that the closed loop system behaves like a lower order system.In recent years, tremendous amount of work has been reported in the literature for controlling nonlinear systems. Control laws based on phase space [9], linear controller design [10], the gain scheduling approach [11], and neural network techniques [12, 13] have been widely used to control nonlinear systems. The theoretical development aspects of sliding mode controller (SMC) are well documented in [14]. In addition to these techniques, backstepping based control design are also utilized to handle nonlinear systems. Backstepping is recursive design methodology for construction of both feedback control laws and lyapunov function in a systematic manner. Nonlinear backstepping designs are strongly related to feedback linearization [15, 16].

In this paper, backstepping strategy embedded with sliding mode control scheme for the continuous time input-affine nonlinear dynamical systems is presented. The proposed controller is based on the sliding mode control and backstepping [17-19]. Here, the backstepping algorithm is modified at the final step by incorporating sliding mode control. The presented methodology derives the controller function in a systematic way to achieve asymptotic regulation of system state vector to their desired trajectories. The control scheme has been implemented for two example systems of input-affine nonlinear dynamical systems category i.e. a single link manipulator system and a ball and beam setup coupled with servomotor.

The organisation of paper is presented as follows. Section I presents the relevance & the general introduction of the paper. Section II describes the sliding mode control methodology with backstepping algorithm to derive the structure of controller for nonlinear systems. The simulation results and analysis of the continuous-time nonlinear affine dynamical systems have been presented in section III. Section IV presents the conclusion of the proposed work. In the end, a brief list of references is given.

II. PROBLEM FORMULATION

In the present section, backstepping based sliding mode control design methodology is proposed for input affine nonlinear class of systems. These results are used in subsequent section to develop controller for example nonlinear systems.

Let us consider a general n^{th} order dynamical system with following description:

$$\begin{aligned} \dot{x}_1 &= F_1(x_1) + G_1(x_1)x_2 \\ \dot{x}_2 &= F_2(x_1,x_2) + G_2(x_1,x_2)x_3 \\ &\vdots \\ \dot{x}_n &= F_n(x_1,x_2,..x_n) + G_n(x_1,x_2,..x_n)u \end{aligned} \tag{1}$$

where $x_1, x_2 \ldots\ldots\ldots\ldots x_n$ represents the states of nth order system, u is scalar control input and $F_i(\underline{x}), G_i(\underline{x})$ for i = 1, 2,…..n-1 are linear functions of the system states where as

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$F_n(\underline{x})$ & $G_n(\underline{x})$ are nonlinear functions. Here $\underline{x}$ is a vector representing system states.

The objective of the controller is to make x_1 track a desired trajectory x_{1d}. The first tracking error is defined as:

$$e_1 = x_1 - x_{1d} \tag{2}$$

The dynamics of e_1 can be described by

$$\dot{e}_1 = F_1(x_1) + G_1(x_1)x_2 - \dot{x}_{1d} \tag{3}$$

then x_{2d} is chosen as virtual control to drive e_1 to zero so that x_1 can track x_{1d}.

$$x_{2d} = G_1^{-1}(-F_1 + \dot{x}_{1d} - K_1 e_1) \tag{4}$$

The above control results in following error dynamics:

$$\dot{e}_1 = -K_1 e_1 \tag{5}$$

Now the next step is to make x_2 track x_{2d}, which insures existence of convergence dynamics (5). So define the second tracking error as

$$e_2 = x_2 - x_{2d} \tag{6}$$

The dynamics of e_2 can be described by

$$\dot{e}_2 = F_2(x_1, x_2) + G_2(x_1, x_2)x_3 - \dot{x}_{2d} \tag{7}$$

To stabilize dynamics in (7), x_{3d} is chosen as virtual control to drive e_2 to zero so that x_2 can track x_{2d}

$$x_{3d} = G_2^{-1}(-F_2 + \dot{x}_{2d} - K_2 e_2) \tag{8}$$

The above control results in following error dynamics:

$$\dot{e}_2 = -K_2 e_2 \tag{9}$$

Proceeding similarly, define (n-1)-th tracking error as

$$e_{n-1} = x_{n-1} - x_{(n-1)d} \tag{10}$$

The corresponding dynamics of the (n-1)-th tracking error can be described by

$$\begin{aligned} \dot{e}_{n-1} &= F_{n-1}(x_1, x_2, ..x_{n-1}) + \\ &G_{n-1}(x_1, x_2, ..x_{n-1})x_n - \dot{x}_{(n-1)d} \end{aligned} \tag{11}$$

where x_{nd} was chosen to drive e_{n-1} to zero.

$$x_{nd} = G_{n-1}^{-1}\left(-F_{n-1} + \dot{x}_{(n-1)d} - K_{n-1}e_{n-1}\right) \tag{12}$$

The above control results in following error dynamics:

$$\dot{e}_{n-1} = -K_{n-1}e_{n-1} \tag{13}$$

in final stage, let the sliding surface is chosen as

$$s = x_n - x_{nd} \tag{14}$$

Then we compute the control input u using SMC design to bring s to zero in finite time. The dynamic of s can be described by

$$\dot{s} = F_n(x_1, x_2, ..x_n) + G_n(x_1, x_2, ..x_n)u - \dot{x}_{nd} \tag{15}$$

The equivalent control is chosen as

$$u_e = G_n(x)^{-1}\left(-F_n(x) + \dot{x}_{nd} - K_n s\right) \tag{16}$$

in the procedure presented through equations (2) - (16), $K_1, K_2 ..... K_n$ are controller parameters and their values are constant to be selected appropriately.

In order to satisfy sliding mode condition given as [13]:

$$\frac{1}{2}\frac{d}{dt}s^2 \le -\eta|s|$$

the following discontinuous switching term is added to equivalent control in (16):

$$u_s = -k\ sgn(s) \tag{17}$$

where sgn(.) is a standard sign function and k is switching gain [13].

The control action then becomes

$$u = u_e + u_s \tag{18}$$

If all the tracking errors $e_1, e_2, e_3 .....e_{n-1}$ are driven to zero as $t \to \infty$, then x_1 will converge to x_{1d}, x_2 will converge to x_{2d} and so on, & hence asymptotic regulation of system states to desired states can be achieved.

III. Simulation Results

In this section, application of proposed approach and detailed simulation results are presented for the nonlinear systems by using sliding mode controller with backstepping. The results are presented for two example systems of proposed class i.e. a single link manipulator system and a ball and beam setup coupled with servomotor.

A. Example 1: Single-link Manipulator

Here, single-link manipulator nonlinear system control problem is explained. The dynamic system equations of a single-link manipulator are given as

$$\begin{aligned} \frac{d\theta}{dt} &= w \\ \frac{dw}{dt} &= -\frac{g}{l}sin\theta + \frac{1}{ml^2}T_a \end{aligned} \tag{19}$$

where θ is angular displacement, w is angular velocity, and T_a is the actuating torque. Let acceleration due to gravity $g = 9.81 m/s^2$; link length l=1m, and link mass m=1kg. Considering $\theta = x_1$, and $w = x_2$ as the state variables, and $T_a = u$ as the control input, the dynamic system equations can be written as

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{g}{l} sin(x_1) + \frac{1}{ml^2} u \qquad (20)$$

The system dynamic in (20) can be written in strict feedback form

$$\dot{x}_1 = F_1(x) + G_1(x) x_2 \qquad (21)$$

$$\dot{x}_2 = F_2(x) + G_2(x) u$$

By selecting

$$F_1(\underline{x}) = 0, \quad G_1(\underline{x}) = 1,$$

$$F_2(\underline{x}) = -\frac{g}{l} sin(x_1), \quad G_2(\underline{x}) = \frac{1}{ml^2}$$

defining the tracking error as

$$e_1 = x_1 - x_{1d} \qquad (22)$$

The time derivative of e_1 is given by

$$\dot{e}_1 = x_2 - \dot{x}_{1d} \qquad (23)$$

Let

$$x_{2d} = \dot{x}_{1d} - K_1 e_1 \qquad (24)$$

where x_{2d} is a desired signal of state variable x_2.

For designing a SMC for the system, switching surface is required to be designed. Let the switching surface *s* be

$$s = x_2 - x_{2d} \qquad (25)$$

where

$$\dot{x}_{2d} = \ddot{x}_{1d} - K_1 \dot{e}_1 \qquad (26)$$

And the derivative of sliding surface *s* is

$$\dot{s} = -\frac{g}{l} sin(x_1) + \frac{1}{ml^2} u - \dot{x}_{2d} \qquad (27)$$

The equivalent control is chosen as

$$u_e = G_2^{-1}(\underline{x})\left(-F_2(\underline{x}) + \dot{x}_{2d} - K_2 s\right) \qquad (28)$$

$$= \left(\frac{1}{ml^2}\right)^{-1} \left(\frac{g}{l} \sin(x_1) + \dot{x}_{2d} - K_2 s\right)$$

where K_1 and K_2 are controller parameters and there values are constant to be selected, appropriately.
To the equivalent control in (28) a switching term discontinuous across the surface

$$u_s = -k \ sgn(s) \qquad (29)$$

is added to achieve controller given in (18).

To avoid the chattering effect caused by the discontinuity of the control law across the sliding surface, the sign function is replaced by the saturation function i.e.

$$u_s = -k \ sat(s/\beta) \qquad (30)$$

where β is the thickness of the boundary layer [13]. The control action then becomes

$$u = u_e + u_s \qquad (31)$$

when applied to the nonlinear system (20), it asymptotically stabilizes x_1 and x_2 to their desired values as t→∞.

The simulation responses of sliding mode control with backstepping for single-link manipulator (20) are shown in Figs. 1 to 4. Fig.1 and Fig.2 show the system state trajectories which converge towards the desired trajectories of the system states. Fig.3 shows the control signal (31) with respect to time. Fig.4 shows the system errors (22, 25) with respect to time. The parameters of the controller are selected as $K_1 = 1$ and $K_2 = 20$ with the switching gain k = 0.001. The parameters of the system used are acceleration due to gravity g=9.81m/s^2, link length l=1m and link mass m=1kg. The initial conditions for x_1, x_2 are 0.1 & 0.1 respectively and the simulation is run for 10 seconds with step size as 0.01. The desired value of the system state is assumed as $x_{1d} = \sin(t)$.

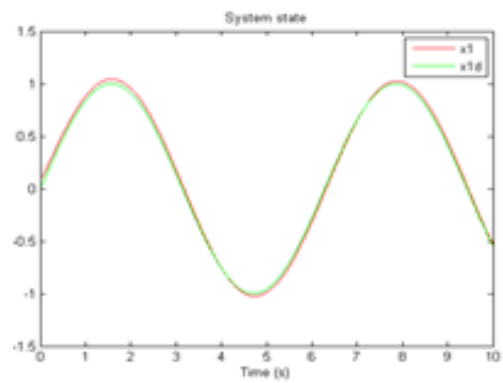

Fig.1 Tracking behaviour of 1st state of system

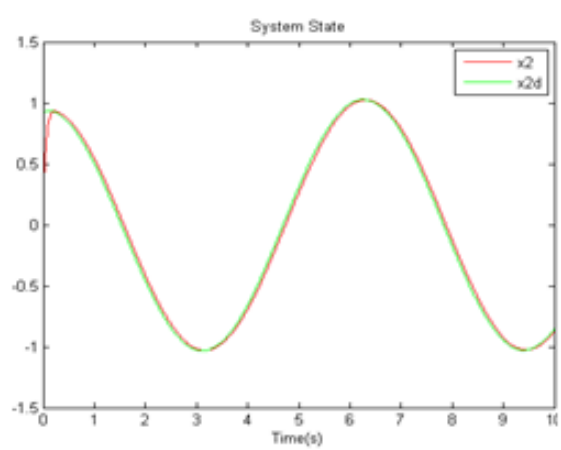

Fig. 2 Tracking behaviour of 2nd state of system

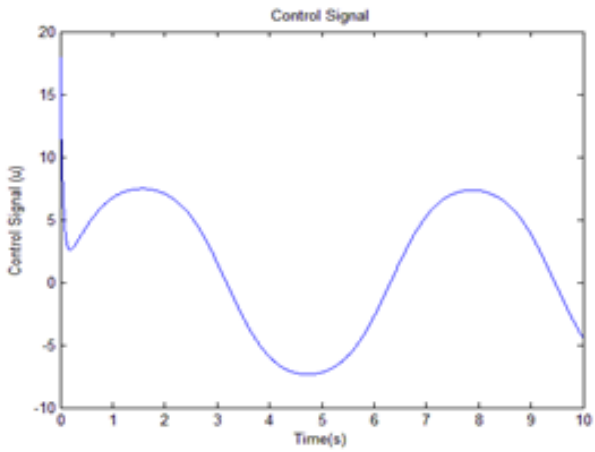

Fig. 3. Behaviour of control signal with respect to time

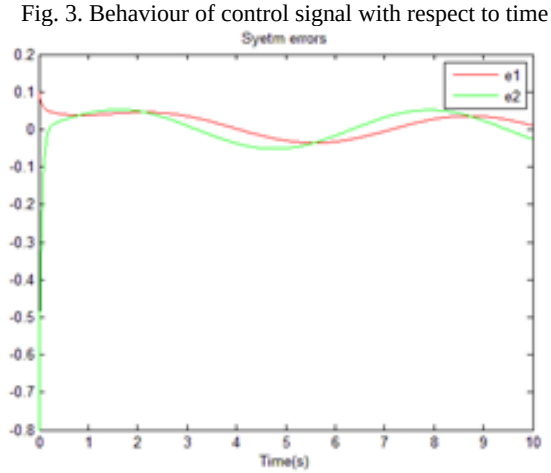

Fig.4. Tracking errors of the system with respect to time

Example 2: Ball and Beam set up coupled with servomotor

In this case, objective is to design controller to position the ball along the track as per desired trajectory by manipulating angular position of servomotor. By selecting ball position (x), ball velocity ($\dot{x}$), servomotor angular displacement (θ) and servomotor angular velocity ($\dot{\theta}$) as state variables $x_1, x_2, x_3, \& \, x_4$ respectively, and taking motor drive voltage (V) as input (u), the above system can be represented by following equations:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= a_1 \sin\left(a_2 x_3\right) \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= a_3 x_4 + a_4 u \end{aligned} \tag{32}$$

with output given as

$$y = x_1 \tag{33}$$

To transform above system coordinates into the new coordinates by using feedback linearization, let the choice of variable be made as

$$\begin{aligned} z_1 &= x_1 \\ z_2 &= x_2 \\ z_3 &= a_1 \sin\left(a_2 x_3\right) \\ z_4 &= a_1 a_2 x_4 \cos\left(a_2 x_3\right) \end{aligned} \tag{34}$$

The dynamic model of the ball and beam system in the new coordinates system can be written as

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= -a_2^2 z_3 x_4^2 + a_3 z_4 + \frac{a_4 z_4}{x_4} u \end{aligned} \tag{35}$$

By selecting

$$F_1\left(\underline{x}\right) = a_1 a_2 \left(-a_2 x_4^2 \sin\left(a_2 x_3\right) + a_3 x_4 \cos\left(a_2 x_3\right)\right)$$

$$G_1\left(\underline{x}\right) = a_1 a_2 a_4 \cos\left(a_2 x_3\right)$$

defining the tracking error as

$$e_1 = x_1 - x_{1d} \tag{36}$$

The time derivative of e_1 is given by

$$\dot{e}_1 = x_2 - \dot{x}_{1d} \tag{37}$$

Let

$$x_{2d} = \dot{x}_{1d} - K_1 e_1 \tag{38}$$

where x_{2d} is a desired signal of state variable x_2.

The time derivative of $e_2 = x_2 - x_{2d}$ is given by

$$\dot{e}_2 = a_1 \sin\left(a_2 x_3\right) - \dot{x}_{2d} \tag{39}$$

Let

$$x_{3d} = \dot{x}_{2d} - K_2 e_2 \tag{40}$$

where x_{3d} is a desired signal of state variable x_3.

The time derivative of $e_3 = a_1 \sin\left(a_2 x_3\right) - x_{3d}$ is given by

$$\dot{e}_3 = a_1 a_2 x_4 \cos\left(a_2 x_3\right) - \dot{x}_{3d} \tag{41}$$

Let

$$x_{4d} = \dot{x}_{3d} - K_3 e_3 \tag{42}$$

where x_{4d} is a desired signal of state variable x_3.

For designing a SMC for the system, switching surface is required to be designed. Let the switching surface s be

$$s = a_1 a_2 x_4 \cos\left(a_2 x_3\right) - x_{4d} \tag{43}$$

where

$$\dot{x}_{4d} = \ddot{x}_{3d} - K_3 \dot{e}_3 \tag{44}$$

The equivalent control is chosen as

$$u_e = G_1^{-1}(\underline{x})\left(-F_1(\underline{x}) + \dot{x}_{4d} - K_4 s\right) \tag{45}$$

where K_1, K_2, K_3 and K_4 are controller parameters and there values are constant to be selected, appropriately.
To the equivalent control in (45), a switching term discontinuous across the surface

$$u_s = -k \ sgn(s) \tag{46}$$

is added as per (17) to achieve controller given in (18).

To avoid the chattering effect caused by the discontinuity of the control law across the sliding surface, the sign function is replaced by the saturation function as given in (30), the final control function is obtained by adding different control terms as per (31).
The application of the controller to the nonlinear system (32), asymptotically stabilizes x_1, x_2, x_3 and x_4 to their desired values as t→∞.

The simulation responses of sliding mode control with backstepping for ball and beam setup coupled with servomotor (32) are shown in Figs. 5 to 10. Fig.5 to Fig.8 show the system state trajectories which converge towards the desired trajectories of the system states. Fig.9 shows the control signal with respect to time. Fig.10 shows the system errors with respect to time. The parameters of the controller are selected as $K_1 = \frac{\pi}{10}$, $K_2 = 6.5, K_3 = \tan\left(\frac{\pi}{4}\right)$ and $K_4 = 1$ with the switching gain k = 0.001. The parameters of the system used are $a_1 = 15, a_2 = 0.06, a_3 = -38.23$ and $a_4 = 71.21$. The initial conditions for x_1, x_2, x_3 and x_4 are 1, 0, $\pi/2$ and 1, respectively and the simulation is run for 25 seconds with step size as 0.01. The desired value of the system state is assumed as $x_{1d} = 0.2\sin(t)$.

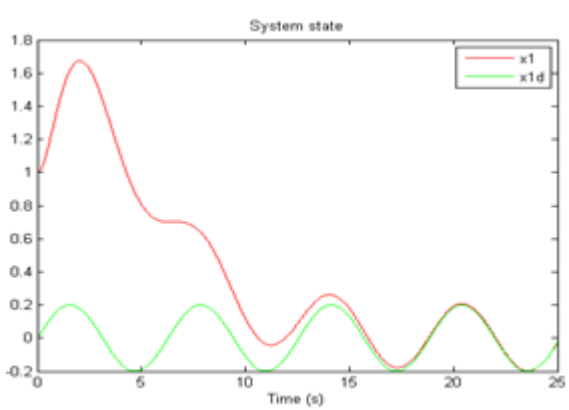

Fig. 5. Tracking behaviour of 1st state of system

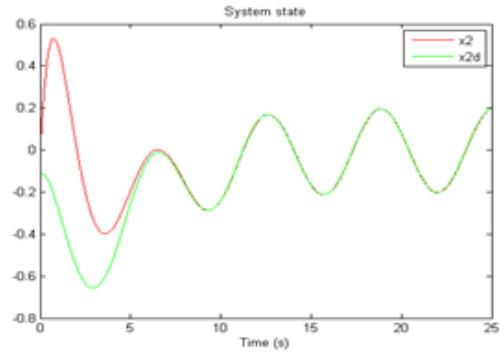

Fig. 6. Tracking behaviour of 2nd state of system

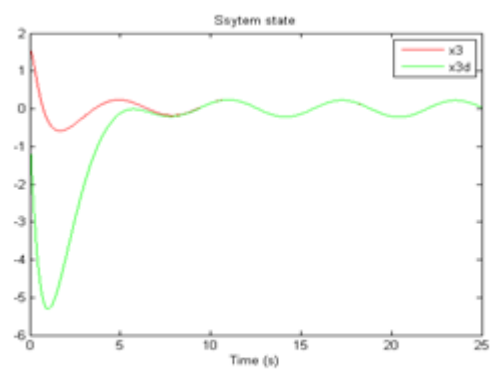

Fig. 7. Tracking behaviour of 3rd state of system

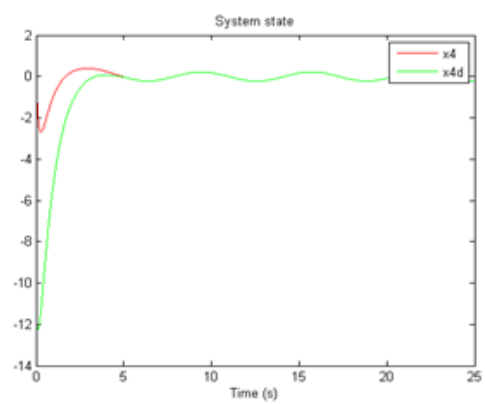

Fig. 8. Tracking behaviour of 4th state of system

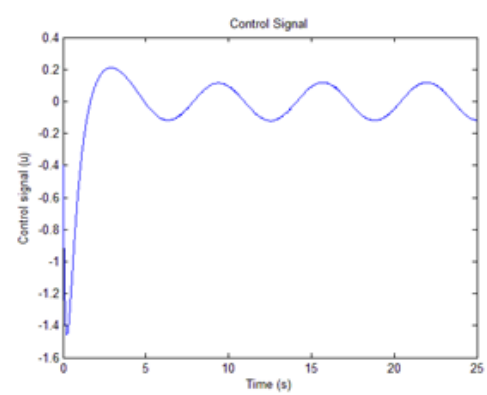

Fig. 9. Behaviour of control signal with respect to time

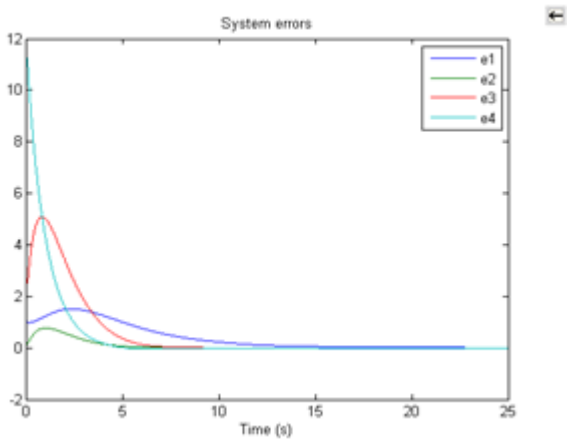

Fig. 10. Tracking errors of the system with respect to time.

IV. Conclusion

In this paper, backstepping strategy with sliding mode control scheme for continuous-time nonlinear input-affine dynamical systems is presented. Sliding mode control is an effective approach for the robust control of class of nonlinear systems with uncertainties. The backstepping strategy has been emphatically combined with sliding mode control methodology to derive the structure of controller. Here, the control scheme is implemented for state tracking problem of two examples of continuous-time affine nonlinear dynamical systems. The simulation results and analysis, justify the effectiveness of the proposed control scheme and asymptotic regulation of system states to their desired trajectories is achieved.

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