About Cyclic Regularity and Proximity Points

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Abstract**—** This work is devoted to self-mappings through a simple formal development to bring together the approximate best proximity points of cyclic self-mappings, the approximate best proximity property and the cyclic asymptotic regularity. A formal development is given which brings together the concepts of approximate best proximity points of cyclic self-mappings, approximate best proximity point property and cyclic asymptotic regularity of cyclic self-mappings. For most of the presented results, special properties of the sets, beyond being nonempty, are not formally required.

Keywords**—**Approximate best proximity points, approximate partial best proximity property, cyclic asymptotic regularity.

I. INTRODUCTION

FIXED point theory is receiving important attention in the last decades because of its applicability to many problems like stability and stabilization of dynamic systems, convergence of sequences, approximation of solutions of algebraic systems, properties of boundedness, convergence and acceleration of convergence of iterative schemes , estimation theory and others. A crucial issue of the developed framework is that many of the relevant properties are linked to the existence of nonempty balls around at least one of the points of the set X of the metric space (X, d) being defined by points x and their images through the involved self-mappings, as well as to the asymptotic regularity of such self-mappings. For many of the presented results, special properties of the sets, beyond being nonempty, are not formally required since the formal development is focused on the study of the existence of approximate fixed points. It is not required either that the metric space be complete by the same reasons. This paper extends such a formalism to 2-cyclic self-mappings. In this way, a simple formal development is given which brings together the concepts of approximate best proximity points of 2-cyclic self-mappings, approximate best proximity (respectively, partial best proximity) point property and cyclic asymptotic regularity of 2-cyclic self-mappings. The extensions of all the obtained results to cyclic self-maps defined on any finite union of subsets of X is direct and are not discussed in the paper although they can be obtained without a small effort from the case of the 2-cyclic case discussed in this paper.

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II. FORMULATION

Let (X, d) be a metric space and let $f: A\cup B \to A\cup B$ be a 2-cyclic self-mapping on the union of two nonempty subsets A and B of X. Since there are only two subsets involved, the self-mapping will be referred to simply as a cyclic self-mapping. The following definition will be then used:

Definitions 2.1. Let (X, d) be a metric space and let $f: A\cup B \to A\cup B$ be a 2-cyclic self-mapping on the union of two nonempty subsets A and B of X. Then

(1) $x \in A\cup B$ is a ε-best proximity point of f (in A or in B) for a given $\varepsilon \in \boldsymbol{R}_{0+}$ if $d(x, fx) \le D+\varepsilon$, where $D = d(A, B) = \inf_{x\in A,\, y\in B} d(x, y)$.

(2) $x \in A$ is a ε-best proximity point of f in A for a given $\varepsilon \in \boldsymbol{R}_{0+}$ if $d(x, fx) \le D+\varepsilon$.

It turns out that $x \in A\cup B$ is an ε- best proximity point of f if and only if

$x \in BP_{\varepsilon}(f) = \{x \in A\cup B : d(x, fx) \le D+\varepsilon\}$

Also $x \in A$ is an ε- best proximity point of f in A if and only if

$x \in BP_{A\varepsilon}(f) = \{x \in A : d(x, fx) \le D+\varepsilon\}$

The following results hold:

Result 2.2. Let (X, d) be a metric space and let $f: A\cup B \to A\cup B$ be a 2-cyclic self-mapping on the union of two nonempty subsets A and B of X. Then, $x \in A\cup B$ is an ε-best proximity point of f then it is an ε_1- best proximity point of f for any real $\varepsilon_1 \ge \varepsilon$.

Since all the subsequent developments are given for cyclic self-mappings on the union of two nonempty subsets of X, we will refer 2-cyclic self-mappings simply as cyclic self-mappings for the sake of notational simplicity.

Result 2.3. Let (X, d) be a metric space and let $f: A\cup B \to A\cup B$ be a 2-cyclic self-mapping on the union of two bounded nonempty subsets A and B of X. Then $x \in A\cup B$ is an ε-best proximity point of f then fx is an ε'- best proximity point for some $\varepsilon' \in \boldsymbol{R}_{+}$.

Definitions 2.4. Let (X, d) be a metric space and let $f: A\cup B \to A\cup B$ be a 2-cyclic self-mapping on the union of two nonempty subsets A and B of X. Then,

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(1) $f: A\cup B\to A\cup B$ has the approximate best proximity point property if $BP_{\varepsilon}(f)\neq\emptyset$ for all $\varepsilon\in\boldsymbol{R}_{0+}$.

(2) $f: A\cup B\to A\cup B$ has the approximate best proximity point property in A if $BP_{A\varepsilon}(f)\neq\emptyset$ for all $\varepsilon\in\boldsymbol{R}_{0+}$.

(3) $f: A\cup B\to A\cup B$ has the ε_0-partial approximate best proximity point property if $BP_{\varepsilon}(f)\neq\emptyset$ for all real $\varepsilon\geq\varepsilon_0$ and a given $\varepsilon_0\in\boldsymbol{R}_{0+}$.

(4) $f: A\cup B\to A\cup B$ has the ε_0-partial approximate best proximity point property in A if $BP_{A\varepsilon}(f)\neq\emptyset$ for all real $\varepsilon\geq\varepsilon_0$ and a given $\varepsilon_0\in\boldsymbol{R}_{0+}$.

(5) $f^2: A\cup B\to A\cup B$ has the ε_0-partial approximate fixed point property if $FP_{\varepsilon}(f)=\left\{x\in A\cup B: d\left(x, f^2x\right)\leq\varepsilon\right\}\neq\emptyset$; $\forall\varepsilon(\in\boldsymbol{R}_+)$ for some given $\varepsilon_0\in\boldsymbol{R}_{0+}$.

(5) $f^2: A\cup B\to A\cup B$ has the ε_0-partial approximate fixed point property in A if $FP_{A\varepsilon}(f)=\left\{x\in A: d\left(x, f^2x\right)\leq\varepsilon\right\}\neq\emptyset$; $\forall\varepsilon(\in\boldsymbol{R}_+)$ for some given $\varepsilon_0\in\boldsymbol{R}_{0+}$.

(7) $f^2: A\cup B\to A\cup B$ has the approximate fixed point property if it has 0-partial approximate fixed point property.

Note that if $A\cap B\neq\emptyset$, and then $D=0$, then the approximate best proximity point property of $f: A\cup B\to A\cup B$ is equivalent to its approximated fixed point property. Note also that $f: A\cup B\to A\cup B$ has the approximate best proximity point property if it has the 0-partial approximate best proximity point property.

Definitions 2.5. Let (X, d) be a metric space and let A and B be nonempty subsets of X with $d(A, B)=D$

Then,

(1) $f: A\cup B\to A\cup B$ is cyclic asymptotically regular if it is cyclic and

$$d\left(f^n x, f^{n+1}x\right)\to D \text{ as } n\to\infty;\ \forall x\in A\cup B$$

(2) $f: A\cup B\to A\cup B$ is cyclic asymptotically ε_0-regular, respectively, ε_0-regular in A, if it is cyclic and

$$d\left(f^n x, f^{n+1}x\right)\to D+\varepsilon_0 \text{ as } n\to\infty;\ \forall x\in A$$

respectively,

Result 2.6. Let (X, d) be a metric space and let A and B be nonempty subsets of X with $d(A, B)=D$. Then any strictly contractive cyclic self-mapping $f: A\cup B\to A\cup B$ is cyclic asymptotically regular, and equivalently, it has the approximate best proximity point property.

Proof: It is direct since, if $f: A\cup B\to A\cup B$ then

$$d\left(f^2x, fx\right)\leq Kd(fx,x)+(1-K)D \tag{1}$$

for some $K\in[0,1)$; $\forall x\in A\cup B$. Thus, it is cyclic asymptotically regular since

$$D\leq d\left(f^{n+1}x, f^n x\right)\leq K^n d(fx,x)+\left(1-K^n\right)D \tag{2}$$

and $d\left(f^{n+1}x, f^n x\right)\to D$ as $n\to\infty$; $\forall x\in A\cup B$. Also, since $f: A\cup B\to A\cup B$ is cyclic asymptotically regular then there is $n_0=n_0(\varepsilon)>0$ for any given $\varepsilon\in\boldsymbol{R}_+$ such that $D\leq d\left(f^{n+1}x, f^n x\right)\leq D+\varepsilon$ so that $BP_{\varepsilon}(f)\neq\emptyset$; $\forall\varepsilon\in\boldsymbol{R}_+$, $\forall x\in A\cup B$. As a result, $f: A\cup B\to A\cup B$ has the approximate best proximity point property. Equivalently, if $BP_{\varepsilon}(f)\neq\emptyset$

$$\Leftrightarrow\left(D\leq d\left(f^{n+1}x, f^n x\right)\leq D+\varepsilon; \forall\varepsilon\in\boldsymbol{R}_+, \forall x\in A\cup B\right) \tag{3}$$

then $\lim_{n\to\infty} d\left(f^{n+1}x, f^n x\right)=D$; $\forall x\in A\cup B$ so that $f: A\cup B\to A\cup B$ is cyclic asymptotically regular

Claim 2.7. Let (X, d) be a metric space and let A and B be nonempty subsets of X with $d(A, B)=D$. If $f: A\cup B\to A\cup B$ is cyclic asymptotically regular then it has the approximate best proximity point property.

Claim 2.8. Let (X, d) be a metric space and let A and B be nonempty bounded subsets of X with $d(A, B)=D$. If $f^2: A\cup B\to A\cup B$ is asymptotically regular then $f: A\cup B\to A\cup B$ has the ε_0-partial best proximity point property for some threshold $\varepsilon_0\in\boldsymbol{R}_{0+}$ with $\varepsilon_0\leq min(diam A, diam B)$.

Theorem 2.9. Let (X, d) be a complete metric space and let A and B be nonempty closed subsets of X with $d(A, B)=D$. Assume that A is approximatively compact with respect to B. Then $f: A\cup B\to A\cup B$ is cyclic asymptotically regular if and only if $f^2: A\cup B\to A\cup B$ is asymptotically regular.

The proof is organized by concluding step-by-step that:

1) since $BP_{\varepsilon}(f)\neq\emptyset; \forall\varepsilon\in\boldsymbol{R}_{0+}$, then $f: A\cup B\to A\cup B$ has 0-best proximity points in A and in B for any $\varepsilon\in\boldsymbol{R}_{0+}$ so that, in particular, $BP_0(f)\neq\emptyset$ and $f: A\cup B\to A\cup B$ has 0-best proximity points. Since A is approximatively compact with respect to B, the set $\{y\in B: d(x, A)=D\}$ is nonempty and, also, if $d(y, x_n)\to d(y, A)=D$ for some $y\in B$ and some sequence $\{x_n\}\subset A$, then there is a convergent subsequence $\{x_{n_k}\}\subset A$ of $\{x_n\}$ and that

2)

$$0<\varepsilon\leq\liminf_{k\to\infty} d\left(f^{2n_k+2}x, f^{2n_k}x\right)\leq\liminf_{k\to\infty} d\left(f^{2n_k+2}x, \lim_{k\to\infty} f^{2n_k}x\right)$$

$$=\liminf_{k\to\infty} d\left(\lim_{k\to\infty} f^{2n_k+2}x, \lim_{k\to\infty} f^{2n_k}x\right)=0$$

For $x'\in B$ we can repeat all the above reasoning for $x=fx'\in A$.

3) $f^2 : A \cup B \to A \cup B$ is asymptotically regular and $BP_\varepsilon(f) \neq \varnothing \Rightarrow \left[F_\varepsilon\left(f^2\right) = \left\{x \in A \cup B : d\left(x, f^2 x\right) \leq \varepsilon\right\} \neq \varnothing\right]$ for any given $\varepsilon \in \boldsymbol{R}_{0+}$ if $f : A \cup B \to A \cup B$ is cyclic asymptotically regular.

4) the converse implication is proved, that is, if $f^2 : A \cup B \to A \cup B$ is asymptotically regular then $f : A \cup B \to A \cup B$ is cyclic asymptotically regular, equivalently,

$$\left[F_\varepsilon\left(f^2\right) = \left\{x \in A \cup B : d\left(x, f^2 x\right) \leq \varepsilon\right\} \neq \varnothing\right] \Rightarrow BP_\varepsilon(f) \neq \varnothing \tag{4}$$

for any given $\varepsilon \in \boldsymbol{R}_{0+}$

or, equivalently, we prove its equivalent contrapositive logic proposition, that is, $BP_\varepsilon(f) = \varnothing \Rightarrow F_\varepsilon\left(f^2\right) = \varnothing$ for any given $\varepsilon \in \boldsymbol{R}_{0+}$. Assume on the contrary that $BP_\varepsilon(f) = \varnothing \Rightarrow F_\varepsilon\left(f^2\right) \neq \varnothing$.Then, $d(x, fx) \geq D + \varepsilon$; $\forall x \in A \cup B$ and $d\left(x, f^2 x\right) < \varepsilon_1$ for some $\varepsilon \in \boldsymbol{R}_+$ and some $x \in A \cup B$ and any $\varepsilon_1 \in \boldsymbol{R}_+$; $\forall x \in A \cup B$. Note that, although $BP_\varepsilon(f) = \varnothing$ for $\varepsilon \in \boldsymbol{R}_+$ is being assumed, $BP_0(f) \neq \varnothing$ so that there are $z \in A$ and $fz \in B$ such that $d(z, fz) = D$ since

5) $f : A \cup B \to A \cup B$ has 0-best proximity points in A and in B. As a result, one has for some $x \in A \cup B$:

$$d\left(f^2 x, fx\right) + \varepsilon_1 > d\left(x, f^2 x\right) + d\left(f^2 x, fx\right) \geq d(x, fx) \geq D + \varepsilon \tag{5}$$

6) for the following sequence of points $z \to fz \to f^2 z$ is generated through $f : A \cup B \to A \cup B$. If $f^2 z = z$ then $d\left(f^2 z, fz\right) = d(z, fz) = D$ holds. Assume that $f^2 z \neq z$ with $d\left(z, f^2 z\right) < \varepsilon_1$; $\forall \varepsilon_1 \in \boldsymbol{R}_{0+}$. Thus, it follows from (5),

$$D < d\left(fz, f^2 z\right) \leq d(z, fz) + d\left(z, f^2 z\right) < \varepsilon_1 + D \tag{6}$$

fails for $\varepsilon_1 = 0$ so that again $d\left(f^2 z, fz\right) = d(z, fz) = D$. Thus, it follows that $d\left(f^2 z, fz\right) = d(z, fz) = D$ so that $D + \varepsilon_1 > D + \varepsilon$ from (2) and then $\varepsilon_1 > \varepsilon$. But this constraint fails for the given $\varepsilon \in \boldsymbol{R}_+$ and $0 \leq \varepsilon_1 \leq \varepsilon$ which contradicts that $\varepsilon_1 \in \boldsymbol{R}_+$ is arbitrary.

Remark 2.10. The condition that A is approximatively compact with respect to B in Theorem 2.9 can be changed by B being approximatively compact with respect to A and the theorem remains valid. Note that if A and B are compact then each of them is approximatively compact with respect to each other. Since A and B are assumed closed then it suffices to assume then, in addition, bounded and to maintaining the result without invoking the approximative compactness assumption.

According to Remark 2.10, Theorem 2.9 leads to the subsequent result:

Corollary 2.11. Let (X, d) be a metric space and let A and B be nonempty compact subsets of X. Then $f : A \cup B \to A \cup B$ is cyclic asymptotically regular if and only if $f^2 : A \cup B \to A \cup B$ is asymptotically regular.

Theorem 2.9 and Corollary 2.11 are, respectively, equivalently to the following results:

Theorem 2.12. Assume that A and B are nonempty closed subsets of X where (X, d) be a metric space. Assume also that A is approximatively compact with respect to B. Then $f : A \cup B \to A \cup B$ has the approximate best proximity point property if and only if $f^2 : A \cup B \to A \cup B$ has the approximate fixed point property.

Corollary 2.13. Let (X, d) be a metric space and let A and B be nonempty compact subsets of X. Then $f : A \cup B \to A \cup B$ has the approximate best proximity point property if and only if $f^2 : A \cup B \to A \cup B$ has the approximate fixed point property.

Theorem 2.14. Let (X, d) be a metric space and let A and B be nonempty bounded closed subsets of X with $d(A, B) = D$. Assume that the cyclic self-mapping $f : A \cup B \to A \cup B$ satisfies the contractive condition:

$$d\left(f^2 x, fx\right) \leq Kd(fx, x) + (1 - K)(D + \delta(x)) \tag{7}$$

for $x \in A \cup B$ and some $K \in [0, 1)$, where $\delta(x) = \varepsilon_{0A}$ if $x \in A$ and $\delta(x) = \varepsilon_{0B}$ if $x \in B$. Then, $f : A \cup B \to A \cup B$ is cyclic asymptotically ε_{0A}-regular in A and cyclic asymptotically ε_{0B}-regular in B has both the ε_{0A}-partial best proximity point property in A and the $\varepsilon_{0B}(\neq \varepsilon_{0A})$-partial best proximity point property in B. Also, $f^2 : A \cup B \to A \cup B$ has not the approximate fixed point property and, equivalently, it is not cyclic asymptotically regular.

Corollary 2.15. Let (X, d) be a metric space and let A and B be nonempty bounded subsets of X with $d(A, B) = D$. Assume that there are nonempty closed sets $A' \subseteq A$ and $B' \subseteq B$ and that the cyclic self-mapping $f : A \cup B \to A \cup B$ satisfies the contractive condition:

$$d\left(f^2 x, fx\right) \leq Kd(fx, x) + (1 - K)(D + 2\delta) \tag{8}$$

; $\forall x \in A \cup B$ and some $K \in [0, 1)$ with $0 \leq \delta \leq min(diam A, diam B)$

Then, $f : A \cup B \to A \cup B$ has the ε-partial approximate best proximity point property for any real $\varepsilon \in [\varepsilon_0 , 2\delta]$ and some real $\varepsilon_0 \in \boldsymbol{R}_{0+}$.

Remark 2.16. Note that Corollary 2.15 does not guarantee that $f : A \cup B \to A \cup B$ is cyclic asymptotically ε-regular for $\varepsilon \in [\varepsilon_0 , 2\delta]$ and some $\varepsilon_0 \in \boldsymbol{R}_{0+}$ since one does not conclude that $d\left(f^{n+1}x , f^n x\right)$ converges to a limit , unless $\delta = 0$, $A' = A$ and $B' = B$ even if (X , d) is complete.

III. Notes Of Relevance To Computing

The results given can be relevant to computations related to stability. For instance, the fixed points can be equilibrium points of a discrete dynamic system. The best proximity points are relevant to limit cycles of a trajectory of a dynamic system which alternates sample-by-sample in-between two (for a 2-cyclic map, p for a p-cyclic one) different sets of the state space. In the particular case when the sets intersect, the best proximity become confluent into a fixed point. In the case they do not intersect, the limit trajectory is the sequence of best proximity points. In the case that they are computing uncertainties in the results, a close approach can be developed via approximate best proximity points. The composite p (2 for 2-cyclic self-mappings) -self-mapping defining the solution trajectories defines fixed points in- between each of the involved (closed) subsets defining the self-mapping which are just the sequence defining the limit cycle of the primary self-mapping. The approximate versions in the case of computing/ modeling uncertainties are direct to the light of the above formalism.

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