Amplitude Death in Conjugate Coupled Topological Network

Himesh Handa, Utkarsh Gupta, and B. B. Sharma

***Abstract*—**Coupled oscillator networks are often used as models for complicated systems that consist of interacting component. Coupled oscillators may contain identical or non identical oscillators, however it is generally assumed that oscillators are identical. Control and stabilization aspects of such networks are prominently explored since long. When nonlinear dynamical oscillators are coupled via conjugate variable, an important phenomenon named amplitude death arises. Oscillations of entire system stop as a result of interaction and converge to a fixed point. This paper, addresses the occurrence of amplitude death phenomenon in conjugate coupled identical chaotic oscillators. It is assumed that chaotic oscillators are arranged in topological networks like mesh topology, ring topology and star topology. The phenomenon of amplitude death is observed in these topological networks and is highlighted in terms of certain conditions on coupling strength. Results are illustrated by taking networks of chaotic Rossler oscillators as example

***Keywords*---**Conjugate coupling, Amplitude death, Topological network, Fixed point

I. Introduction

IN pursuit of chaos control, many methods and techniques like open loop strategies, parameter dependent approaches, feedback control and adaptive control work remarkably well up to a point. However, there are still many problems that need to be considered. For instance, the parameter variation control method is out of question when none of the system parameters is accessible. Similarly the feedback control and open loop control methods may not be feasible when an explicit, credible mathematical model cannot be constructed due to extreme complexity of physical system to be controlled. Control of chaotic oscillations and stabilization of unstable dynamics have always been an area of great interest for researchers due to its wide applications in many areas. Irregular oscillations can be undesirable in physical systems. Suppressing irregular oscillations can be useful for various physical systems. When two or more dynamical systems are coupled by some signal the system may synchronize under some conditions [1]. There are various schemes for control and synchronization of different chaotic systems which have been explored by various researchers since long such as back-stepping design technique [2], contraction theory based approach [3], observer based design [4] and control schemes via feedback linearization etc [5], which are primarily used for synchronization. In majority of synchronization schemes, diffusively coupled complex networks are considered. As far as phenomenon of amplitude death is concerned, the concept has been discussed widely in [6-10]. A lot of work has been reported on amplitude death in literature and it is a subject of great interest for research due to its wide applications in variety of fields like stabilization of DC grids [11], and in chemical systems etc [12] [13].

Coupled nonlinear oscillators exhibit different phenomenon in both theoretical and experimental studies depending on manner in which they interact. Coupling may be diffusive or conjugate in nature. Conjugate coupled chaotic oscillators show a novel phenomenon named amplitude death. It is defined as process of complete suppression of oscillations. Oscillations may be irregular or periodic and as a result of interaction the whole system will converge to fixed points depending on coupling type and variation of system parameters. These points may be the fixed points of uncoupled system or the new fixed points which are generated by coupling. Coupled system either converges to fixed points of uncoupled system known as Hop-bifurcation or may converge to new fixed point which is the fixed point of coupled systems known as saddle node bifurcation that eradicates the periodic orbit leading to decay of oscillations. There are number of coupling schemes except conjugate coupling like dynamic coupling [14], time delay interaction [15] and indirect coupling [16], mean field diffusion [17] and environmental coupling [18] which lead to the amplitude death.

The early work in this direction was mainly concerned with amplitude death in two conjugate coupled oscillators by exploring the spectrum of largest Lyapunov exponent with the variation in coupling parameter. Later, analytical conditions in terms of N number of local or global conjugate coupled systems with coupling constant were proposed in [19] to

Himesh Handa, is Assistant Professor with the Electrical Engineering Department National Institute of Technology, Hamirpur H. P.-177005, INDIA, Phone- +919418073361, 01972254546,
email: hklhanda@gmail.com

Utkarsh Gupta is with the Electrical Engineering Department National Institute of Technology, Hamirpur, H. P.-177005, INDIA(e-mail: utkarshnith@gmail.com)

B. B. Sharma is Assistant Professor with the Electrical Engineering Department, National Institute of Technology, Hamirpur, H. P. -177005, INDIA (e-mail: bhushan@nith.ac.in).

theoretically determine the conditions for amplitude death. The increasing interest in amplitude death phenomenon has led many researchers to consider amplitude death in large networks of coupled system with different coupling configurations. The work presented in this paper, considers topological networks like mesh network, ring network and star network for which phenomenon of amplitude death is explored and analytical formulations for these networks is also derived on the basis of exploring spectrum of largest Lyapunov exponent while varying coupling strength. Results are appropriately shown for these networks of chaotic Rossler system to justify the onset of amplitude death. Mesh, ring and star networks are discussed in section III, section IV & section V along with analytical results. Section VI concludes the paper with discussion and summary of results.

II. System Dynamics

For the purpose of investigation of amplitude death phenomenon, chaotic Rossler systems based network is considered. The dynamics of Rossler oscillator is described as:

$$\begin{aligned} \dot{x} &= -y-z \\ \dot{y} &= x+ay \\ \dot{z} &= b+z(x-c) \end{aligned} \tag{1}$$

Where a, b and c are system parameters. It shows oscillatory nature for a wide range of values of parameters a, b and c with a=0.2 ,b=0.2 and c=5.7 , this system exhibits chaotic behaviour. Fig.1 shows the phase portrait of Rossler attractor in 3 dimensions for initial conditions as [0.1 -0.1 0.1].

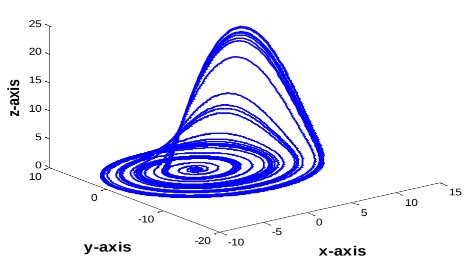

Fig. 1 Phase portrait of Rossler attractor.

III. Conjugate Coupled Mesh Topological Network

In mesh topology, every system has dedicated point to point link with every system. For K number of systems, there are (K-1), input/output ports that are to be connected to (K-1) systems. The traffic problems are reduced to large extent due to dedicated links. There is an advantage of privacy or security also as every message travels along a dedicated line and only the intended recipient could see it. Let us consider a general configuration of mesh connected conjugate coupled network described as:

$$\dot{X}_i = F_i(X_i) + \in G(X_i, X_j^{'}) \tag{2}$$

where coupling term is given as**:**

$$G(X_i, X_j^{'}) = \sum_{j=1,\, j\neq i}^{K} (X_j^{'} - (K-1)X_i) \tag{3}$$

For i=1,2…,K and $F_i(X_i)$ is dynamics of uncoupled ith dynamical system, G is coupling function, X_i is vector of dynamical variables (X_i = x_i, y_i, z_i) corresponding to ith dynamical system, superscript (') is used to indicate conjugate coupled variable and it is assumed that $\in>0$. Let us consider z variable for coupling within mesh connected network. The system dynamics of conjugate coupled mesh network with four Rossler systems is described as:

$$\begin{aligned} \dot{x}_i &= -y_i - z_i + \in\left(\sum_{\substack{j=1\\ j\neq i}}^{4} z_j - 3x_i\right) \\ \dot{y}_i &= x_i + ay_i \\ \dot{z}_i &= b + z_i(x_i - c) \end{aligned} \tag{4}$$

Where i=1,2…,4. Here system parameters are a=0.2, b=0.2 & c=5.0. This type of network topology is common in regional telephone offices that need to be connected to every regional office. Typical mesh topology for a network of four systems is shown in Fig. 2. Linear stability analysis of mesh system described in Eq. (4) about fixed point origin leads to following characteristics equation [19] :

$$\lambda^3 + \lambda^2(4.8+3\in) + 14.4\lambda\in + (5-3\in) = 0 \tag{5}$$

Which is independent of time and depends on coupling strength $\in$. Here, λ indicates eigenvalues of coupled system. System is asymptotically stable if all the eigenvalues of the system have negative real part. So by applying Routh-Hurwitz criterion for the mesh network approximate range of coupling strength $\in$ to which amplitude death occurs comes out as $0.0667 < \in < 1.66$. As we can see from Fig. 3(a) that largest Lyapunov exponent becomes negative at $\in = 0.6$ and remains negative till $\in$=1.66, hence this region confirms the existence of amplitude death. Fig. 3(b) shows that variation in oscillations of state x of coupled systems that converge to fixed point origin which is the fixed point of uncoupled system. It shows that Hopf -bifurcation takes place that leads the behaviour to change from chaotic nature to amplitude death. At $\in = 0.6$, largest Lyapunov exponent becomes most negative. After $\in = 0.6$, largest Lyapunov Exponent starts increasing again and the system tends towards its chaotic nature after $\in >$ 1.66. Fig. 3(c) shows the average distance among trajectories at $\in = 0.6$ which goes to zero after t=10 sec. It clearly confirms the occurrence of amplitude death. Here, only the variation of x states is shown, similar results follow for variation for y and z states. Fig. 3(c) explains the amplitude death in other two states as well. Here, the average distance function is calculated as $d(t) = \frac{1}{K}\sum_{i=1}^{K}\sqrt{x_i^2 + y_i^2 + z_i^2}$ where t is progression time period for coupled system. In the event of amplitude death,

http://dx.doi.org/10.15242/IIE.E0514604

d(t) converges to zero in case of Hopf-bifurcation or converges to a constant value in case of saddle-node bifurcation. Fig. 3(d) explains the monotonic decrease in amplitude of state x, when coupling strength is taken as $\in$ = 1.25.

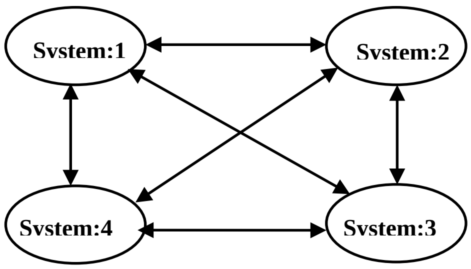

Fig. 2 Graphical representation of mesh topological network for K= 4

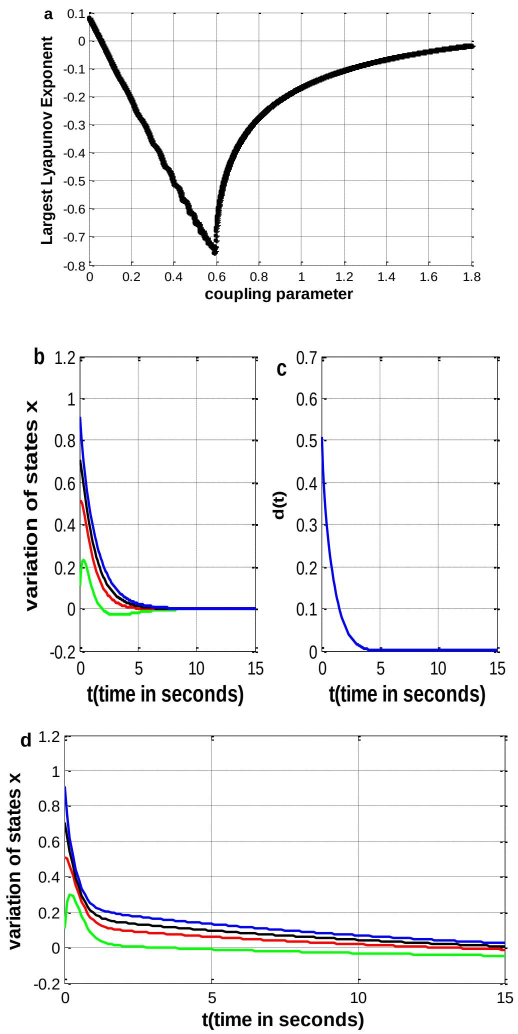

Fig. 3(a) Spectrum of largest Lyapunov exponent as a function of coupling strength $\in$, (b) Transient trajectories of the x components of mesh network as a function of time showing the occurrence of amplitude death at $\in$ =0.6, (c) Plot of average distance among the trajectories as a function of time, (d) Plot of x component of mesh network as a function of time at $\in$=1.25 which shows monotonic decrease in the amplitude.

IV. Conjugate coupled Ring topological network

In ring topology, each system has a dedicated point to point connection with only two systems, i.e. one each on either side of it. The signal is passed along the ring in one direction from one system to other. The representative diagram showing five systems in ring topology is shown in Fig. 4.

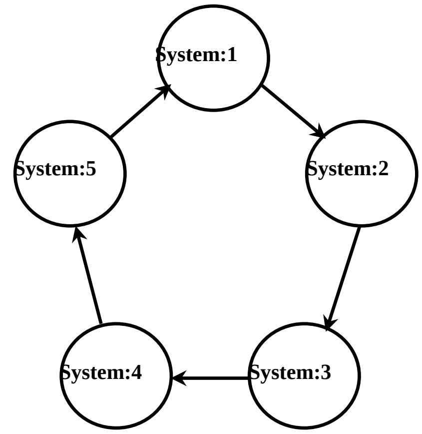

Fig. 4 Graphical representation of ring topological network for K=5.

The generalized system equations for ring connected conjugate coupled network can be written as:

$$\dot{X}_i = F_i\left(X_i\right) + \in G(X_i, X_j') \tag{6}$$

$$G(X_i, X_j') = \begin{bmatrix} (X_j' - X_i) \text{ for } i = 1,2...K\text{-}1 \\ \text{if, } i = K, \text{ then } j = \text{mod}(j, K) \end{bmatrix} \tag{7}$$

where j=i+1, & i≠j. $F_i(X_i)$ is dynamics of uncoupled system corresponding to i^{th} dynamical system and it is assumed that $\in$>0. Here, ring network consisting of five Rossler systems is analyzed and z variable is considered for coupling. The system dynamics of five conjugate coupled Rossler systems in ring configuration is given as:

$$\begin{aligned} \dot{x}_i &= -y_i - z_i + \in (z_j - x_i) \\ \dot{y}_i &= x_i + a\,y_i \\ \dot{z}_i &= b + z_i(x_i - c) \end{aligned} \tag{8}$$

for i= 1,2…,5, j=i+1 & i≠j. System parameter are taken as a=0.2, b=0.2, c=5.7. Linear stability analysis of ring system described in (8) about fixed point origin gives the characteristic equation as [19]:

$$\lambda^3 + \lambda^2(5.5+\in) + \lambda(5.5\in - 1.14) + (5.7 - 1.14\in) = 0 \tag{9}$$

By applying Routh-Hurwitz criteria characteristic equation (9) gives the approximate range of $\in$ for which amplitude death occurs is $0.2 < \in < 5.7$. As we can see in Fig. 5(a) that largest Lyapunov exponent becomes negative at $\in$ = 0.2 and remains negative till $\in$ = 5.7, which confirms the occurrence of amplitude death. Fig. 5(b) shows that variation in oscillations of state x of coupled systems that converge to fixed point origin which is the fixed point of uncoupled system. It shows Hopf- Bifurcation similar to previous case that leads the system from chaotic nature to amplitude death. At $\in$=1.75 largest Lyapunov exponent becomes most negative. After $\in$=1.75 largest Lyapunov exponent starts increasing again and system tends towards its chaotic nature after $\in$>5.7. Fig. 5(c) shows the average distance among trajectories at $\in$=1.75 which goes to zero after t=10 sec which confirms the occurrence of amplitude death. The discussion will remain same here for the variation of other two states. Fig. 5(c) shows the amplitude death in other two states which depict the variation of average distance between the trajectories. The basic difference in this case is that one can achieve the amplitude death for higher values of coupling strength for same settling time as compared to previous case.

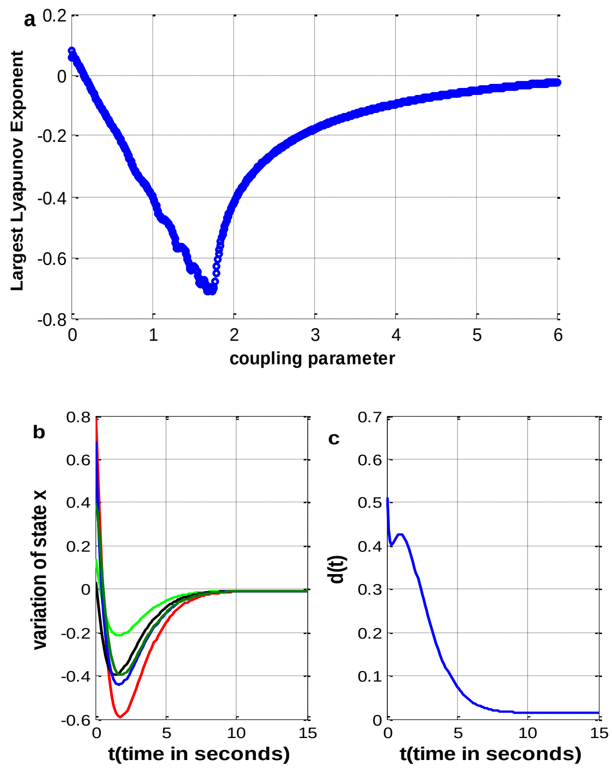

Fig. 5(a) Spectrum of largest Lyapunov exponent as a function of coupling strength $\in$ for ring connected conjugate coupled Rossler system, (b) Transient trajectories of the x components of system in ring network as a function of time showing the occurrence of amplitude death at $\in$ = 1.75, (c) Plot of average distance among the trajectories as a function of time

V. CONJUGATE COUPLED STAR TOPOLOGICAL NETWORK

In star topology, each system has a dedicated point to point link only to central controller. The systems are not directly linked to other systems. The practical example for this topology is high speed LAN's with a central hub.

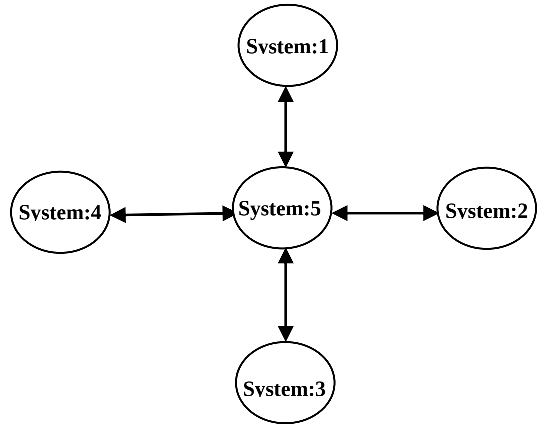

Fig. 6 Graphical representation of Star topological network for K=5.

Let us consider a general configuration of star connected conjugate coupled networks described as:

$$\dot{X}_i = F_i\left(X_i\right) + \in G\left(X_i, X_j^{'}\right) \tag{10}$$

$$G(X_i, X_j^{'}) = \begin{bmatrix} (X_K^{'} - X_i), \text{ for i=[1,K-1]} \\ \sum_{j=1,\, j\neq i}^{K-1} X_j^{'} - (K-1)X_K, \text{ for i=K} \end{bmatrix} \tag{11}$$

for i= 1,2…,K and $F_i(X_i)$ is dynamics of uncoupled i[th] dynamical system, G is coupling function , $\in$ is coupling strength, X_i is dynamical variable $(X_i = x_{i,} y_{i,} z_i)$ corresponding to i[th] dynamical system with coupling strength $\in$ ›0. The system dynamics of five conjugate coupled (K=5) star connected networks is described as follows:

$$\begin{aligned} \dot{x}_i &= -y_i - z_i + \in (z_5 - x_i) \\ \dot{y}_i &= x_i + a\,y_i \qquad \text{for i= [1,4]} \\ \dot{z}_i &= b + z_i(x_i - c) \end{aligned} \tag{12}$$

$$\begin{aligned} \dot{x}_5 &= -y_5 - z_5 + \in (z_1 + z_2 + z_3 + z_4 - 4x_5) \\ \dot{y}_5 &= x_5 + a\,y_5 \\ \dot{z}_5 &= b + z_5(x_5 - c) \end{aligned} \tag{13}$$

The system parameter are taken as a=0.2,b=0.2 and c=5.7. The linear stability analysis of star system described in (12) & (13) about fixed point origin gives the characteristics equation as [19]:

$$\lambda^3 + \lambda^2(5.5+\in) + \lambda(5.5\in - 1.14) + (5.7 - 1.14\in) = 0 \tag{14}$$

which depends on coupling strength and eigenvalues of coupled system. So by applying Routh-Hurwitz criterion the approximate range of coupling strength $\in$ for which amplitude death occurs is $0.2\langle \in \langle 5.7$ which is same as for the ring network but the variation in spectrum of largest Lyapunov exponent is different for this case. The basic reason for the same range of amplitude death as previous case is the coupling with z variable because of which the basic mathematical structure remains same except for one system dynamics. When

http://dx.doi.org/10.15242/IIE.E0514604

the Jacobian matrix is formulated & corresponding the characteristics equation is obtained. The terms for z variable part becomes zero that's why here, we get the same characteristics equation as for previous case. But if we couple it via y-variable, then different results are obtained. We can see from fig. 7(a) that largest Lyapunov exponent becomes negative at $\in$ =0.2 and remains negative till $\in$ =5.7. It confirms the occurrence of amplitude death. Fig. 7(b) shows that variation in oscillations of state x of coupled systems that converge to fixed point origin which is the fixed point of uncoupled system. It shows that Hopf- Bifurcation takes place that leads the system behaviour from chaotic nature to amplitude death. At $\in$=0.85 largest Lyapunov exponent becomes most negative. After $\in$=0.85 largest Lyapunov Exponent starts increasing again and the system tends towards its chaotic nature after $\in$ ›0.85. Fig. 7(c) confirms the occurrence of amplitude death. It takes more time to converge as compare to previous case.

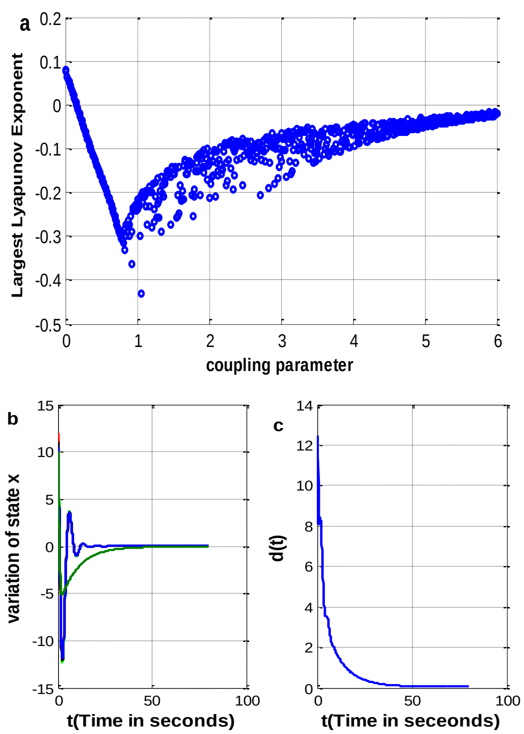

Fig. 7(a) Spectrum of largest Lyapunov exponent as a function of coupling strength $\in$ for star connected conjugate coupled Rossler system, (b) Transient trajectories of the x components of star network as a function of time showing the occurrence of amplitude death at $\in$ = 0.85, (c) Plot of average distance among the trajectories as a function of time.

VI. Conclusion

In this paper, the phenomenon of amplitude death in topological networks like ring network, mesh network and star network for conjugate variable coupling is presented and corresponding bounds on coupling strength are established analytically. The results are verified numerically by considering Rossler oscillator based networks. Amplitude death can be essential and coveted feature in variety of fields. Although, there has been significantly great work on amplitude death but many challenges remain both at fundamental as well as application level. The study can also be extended for conjugate coupled systems with more complex topology. The methodology can be extended to address amplitude death in hyperchaotic systems with various coupling schemes like dynamic coupling, time delay coupling etc.

References

[1] L. M. Pecora, and T. L. Carroll, "Synchronizing chaotic circuits," IEEE Trans Circ Syst. I Vol 38, pp. 453-56, 1991.

[2] X. Tan, J. Zhang and Y. Yang, "Synchronizing chaotic systems using backstepping design," Chaos Solitons and Fractal, Vol. 16, pp. 37-45, 2003. http://dx.doi.org/10.1016/S0960-0779(02)00153-4

[3] B. B. Sharma, I. N. Kar, "Contraction theory based adaptive synchronization of chaotic systems," Chaos, Solitons and Fractals., Vol. 41, pp. 2437-2447, 2009. http://dx.doi.org/10.1016/j.chaos.2008.09.031

[4] B. B. Sharma, I. N. Kar, "Observer- based synchronization schemes for a class of chaotic systems using contraction theory," Nonlinear Dynamics, Vol. 63, pp. 429-445, September, 2010. http://dx.doi.org/10.1007/s11071-010-9813-4

[5] B. B. Sharma, I. N. Kar,"Parametric convergence and control of chaotic system via adaptive feedback linearization", Chaos Solitions and Fractls, Vol. 40, pp. 1475-1483, 2009. http://dx.doi.org/10.1016/j.chaos.2007.09.060

[6] R. Karnatak, R. Ramaswamy, and Awadesh Prasad, "Amplitude Death in the Absence of time delays in identical coupled oscillators," Physical Review, Vol. E76, pp. 035201- 4, 2007.

[7] Awadesh Prasad, "Amplitude death in coupled chaotic oscillators," Physical Review, Vol. E72, pp. 056204-10, 2005.

[8] Awadesh Prasad, "Time varying interaction leads to amplitude death in coupled nonlinear oscillators," Pramana-Journal of Physics, pp. 1-9, 2013.

[9] K. Konishi, "Amplitude death induced by dynamic coupling," Physical Review Letters, Vol. E68, pp. 067202-4, 2003.

[10] K. Konishi, "Limitation of time-delay induced amplitude death," Physics Review Letters, Vol. A 341, pp. 401-409, 2005.

[11] S. R. Huddy and J. D. Sufca, "Amplitude death solutions for stabilization of dc microgrids with instantaneous constant power loads" IEEE Transcation, Power Electronics,Vol. 28, pp.247-253, 2013. http://dx.doi.org/10.1109/TPEL.2012.2196056

[12] F. M. Atay, "Distributed delay facilitate amplitude death of coupled oscillators " Physical Review Letters, Vol. 91, pp. 094101-094104, 2003. http://dx.doi.org/10.1103/PhysRevLett.91.094101

[13] M. Yoshimoto, "Phase death mode in two coupled chemical oscillators", Chem. Phys. Lett. Vol. 280 ,pp. 539-543, 1997. http://dx.doi.org/10.1016/S0009-2614(97)01175-5

[14] K. Konishi, "Amplitude death induced by a global dynamic coupling," International Journal of Bifurcation and chaos , Vol. 17, pp. 2781-2789, 2007. http://dx.doi.org/10.1142/S0218127407018750

[15] K. Konishi, " Amplitude death in oscillations coupled by a one way ring time delay connection," Physical Review, Vol. 70, pp. 066201-066206, 2004.

[16] Awadesh Prasad, P. R. Sharma and M. Shrimali, " Ampltude death in nonlinear oscillations with indirect coupling," Physics Letters Vol. A 376, pp. 1562-1566, 2012. http://dx.doi.org/10.1016/j.physleta.2012.03.033

[17] Awadesh Prasad and M. D. Shrimali, "Amplitude death with mean field diffusion," Physical Review Vol. E 85, pp. 057204-057207, 2012. http://dx.doi.org/10.1103/PhysRevE.85.057204

[18] V. Resmi, G. Ambika, R.E. Amritkar and G. Rangarajan, " Amplitude death in complex networks induced by environment, " Physics Review, Vol. E. 84, pp. 046211-046217, 2012 .

[19] X. Zhang, Y. Wu, and J. Peng, "Analytical conditions for amplitude death induced by conjugate variable couplings," International journal of Bifurcation and Chaos, Vol. 21, pp. 225-235, 2010. http://dx.doi.org/10.1142/S0218127411028386