Bounds for Laplacian Energy Of Binary Labeled Graph

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Abstract— Let G be a binary labeled graph and $A_l(G)$ =(l_{ij}) be its label adjacency matrix. For a vertex v_i, we define label degree as $L_i = \sum_{i=1}^{n} l_{ij}$. In this paper, we define Label Laplacian energy $LE_l(G)$. It depends on the underlying graph G and labels of the vertices. We obtain some results on label Laplacian spectrum. We also obtain some bounds for label Laplacian energy.

Keywords—Label Laplacian Matrix, Label Laplacian Eigenvalues, Label Laplacian Energy.

I. Introduction

LET G be a graph of order n. The energy of the graph G was first defined by Gutman [8] in 1978 as the sum of the absolute eigenvalues of G. It represents a proper generalization of a formula valid for the total π-electron energy of a conjugated hydrocarbon as calculated by the Huckel molecular orbital (HMO) method in quantum chemistry. For recent mathematical work on the energy of a graph see ([3]-[6], [10], [14]). In connection with graph energy, energy -like quantities were also considered for other matrices: Laplacian [7], distance [9], minimum covering [1], label matrix[13] etc.

In 2013, P.G. Bhat and S. D'Souza [13] have introduced a new matrix $A_l(G)$ called label matrix of a binary labeled graph G = (V,X), whose elements are defined as follows:

$$l_{ij} = \begin{cases} a, & \text{if } v_i v_j \in X(G) \text{ with } l(v_i) = l(v_j) = 0, \\ b, & \text{if } v_i v_j \in X(G) \text{ with } l(v_i) = l(v_j) = 1 \\ c, & \text{if } v_i v_j \in X(G) \text{ with } l(v_i) = 0 \text{ and } l(v_j) = 1 \text{ or vice versa} \\ 0, & \text{otherwise} \end{cases}$$

where a, b, and c are distinct non zero real numbers. The eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of $A_l(G)$ are said to be label eigenvalues of the graph G and form its label spectrum. The label eigenvalues satisfy the following simple relations:

$$\sum_{i=1}^{n} \lambda_i = 0 \qquad and \qquad \sum_{i=1}^{n} \lambda_i^2 = 2Q$$

Where $Q = n_1 a^2 + n_2 b^2 + n_3 c^2$ and n_1, n_2 and n_3 denote number of edges with (0,0), (1,1) and (0,1) as end vertex labels respectively. The *label degree* of the vertex v_i, denoted by L_i, is given by $L_i = \sum_{i=1}^{n} l_{ij}$. A Graph G is said to be *k-label regular* if L_i = k for all i. The label Laplacian matrix of a binary labeled graph G is defined as

$$L_l(G) = \mathrm{Diag}(L_i) - A_l(G)$$

where Diag(Li) denotes the diagonal matrix of the label degrees. Since $L_l(G)$ is real symmetric, all its eigenvalues μ_i, i = 1,2,...,n, are real and can be labeled as $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$ These form the *label Laplacian spectrum* of G. Several results on Laplacian of Graph G are reported in the Literature ([5, 10, 11, 12, 15]) This paper is organized as follows. In the next section we establish some general results on Laplacian Label eigenvalues μi. In the following section lower bound and upper bounds for $LE_l(G)$ are obtained.

II. Label Laplacian Energy

The following Lemma 2.0.1 shows the similarities between the spectra of label matrix and label Laplacian matrix. For a labeled graph, let $P_A(x)$ and P(x) denote the label and label Laplacian characteristic polynomials respectively.

Lemma 2.0.1. If $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is the label spectrum of k-label regular graph G, then $\{k-\lambda_n, k-\lambda_{n-1}, \ldots, k-\lambda_1\}$ is the label Laplacian spectrum of G.

Proof. The label Laplacian characteristic polynomial for k-label regular graph G is given by

$P_L(x)=\det(L_l(G)-xI)=(-1)^n\det(A_l(G)-(k-x)I)=(-1)^n P_A(k-x)$.

Thus, if $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ is the label spectrum of k-label regular graph G, then from equation 2.1, it follows that $k-\lambda_n \geq k-\lambda_{n-1} \geq \ldots \geq k-\lambda_1$ is the label Laplacian spectrum of G.

We first introduce the auxiliary eigenvalues γ_i, defined as

$$\gamma_i = \mu_i - \frac{1}{n}\sum_{i=1}^{n} L_i$$

Lemma 2.0.2. If $\{\mu_1, \mu_2, \ldots, \mu_n\}$ are the label Laplacian eigenvalues of $L_l(G)$, then $\sum_{i=1}^{n} \mu_i^2 = 2Q + \sum_{i=1}^{n} L_i^2$

Lemma 2.0.3. Let G be a binary labeled graph of order n. Then

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$\sum_{i=1}^{n} \gamma_i = 0$ *and* $\sum_{i=1}^{n} \lambda_i^2 = 2R$ where

$$R = Q + \frac{1}{2}\sum_{i=1}^{n}\left(L_i - \frac{1}{n}\sum_{i=1}^{n} L_j\right)^2$$

Let G be a binary labeled graph of order n. then the label Laplacian energy of G, denoted by $LE_l(G)$, is defined as $\sum_{i=1}^{n} |\gamma_i|$ i.e. $LE_l(G) = \sum_{i=1}^{n}\left|\mu_i - \frac{1}{n}\sum_{i=1}^{n} L_i\right|$

In 2006, I. Gutman and B. Zhou defined Laplacian energy LE(G) of a graph G. More on Laplacian energy reader can refer ([7], [14], [16], [17]).

Lemma 2.0.4. If G is k- label regular, then $LE_{l}(G) = E_l(G)$

III. Bounds For The Label Laplacian Energy

Lemma 3.0.5. [16] Let $a_1, a_2, \ldots, a_n$ be non-negative numbers. Then

$$n\left[\frac{1}{n}\sum_{i=1}^{n} a_i - \left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}}\right] \le n\sum_{i=1}^{n} a_i - \left(\sum_{i=1}^{n}\sqrt{a_i}\right)^2$$

$$\le n(n-1)\left[\frac{1}{n}\sum_{i=1}^{n} a_i - \left(\prod_{i=1}^{n} a_i\right)^{1/n}\right]$$

Theorem 3.1. Let G be a binary labeled graph with n vertices and m edges. Then

$$\sqrt{2R + n(n-1)\Delta^{\frac{2}{n}}} \le LE_l(G) \le \sqrt{2(n-1)R + n\Delta^{\frac{2}{n}}},$$

Where, $\Delta = \left|\det\left(L_l(G) - \frac{1}{n}\sum_{j=1}^{n} L_j I\right)\right|$

Proof: Note that

$\sum_{i=1}^{n} |\gamma_i| = LE_l(G)$ and $\sum_{i=1}^{n} \lambda_i^2 = 2R$

Using Lemma 3.0.5, it can be easily checked that Theorem 3.1 is true if $\Delta=0$.

Now we assume that $\Delta \neq 0$.

By setting $a_i = \gamma_i^2$, i=1,2,…..,n

and $K = n\left[\frac{1}{n}\sum_{i=1}^{n}\gamma_i^2 - \left(\prod_{i=1}^{n}\gamma_i^2\right)^{\frac{1}{n}}\right] \ge 0,$

From Lemma 3.0.5, we have

$$K \le n\sum_{i=1}^{n}\gamma_i^2 - \left(\prod_{i=1}^{n}|\gamma_i|\right)^2 \le (n-1)K$$

Which can be further expressed as

$$K \le 2nR - (LE_l(G))^2 \le (n-1)K$$

$$K = n\left[\frac{1}{n}\sum_{i=1}^{n}\gamma_i^2 - \left(\prod_{i=1}^{n}\gamma_i^2\right)^{\frac{1}{n}}\right]$$

$$= n\left[\frac{1}{n}2R - \Delta^{\frac{2}{n}}\right] = 2R - n\Delta^{\frac{2}{n}}$$

By substituting in above inequality, we obtain

$$\sqrt{2R + n(n-1)\Delta^{\frac{2}{n}}} \le LE_l(G) \le \sqrt{2(n-1)R + n\Delta^{\frac{2}{n}}}.$$

Theorem 3.2. Let G be a binary labeled graph of order $n \ge 2$. Then $2\sqrt{R} \le LE_l(G) \le \sqrt{2nR}$

Proof: Consider the sum

$$S = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(|\gamma_i| - |\gamma_j|\right)^2$$

$$= 2n\sum_{i=1}^{n}|\gamma_i|^2 - 2\left(\sum_{i=1}^{n}|\gamma_i|\right)\left(\sum_{j=1}^{n}|\gamma_j|\right)$$

$$= 2n.2R - 2(LE_l(G))^2$$

$$= 4nR - 2(LE_l(G))^2$$

Note that $S \ge 0$ i.e. $4nR - 2(LE_l(G))^2 \ge 0$

Which implies $LE_l(G) \le \sqrt{2nR}.$

Also we have $\left(\sum_{i=1}^{n}\gamma_i\right)^2 = 0$ and the fact that $R \ge 0.$

$$\sum_{i=1}^{n}\gamma_i^2 = \left(\sum_{i=1}^{n}\gamma_i\right)^2 - 2\sum_{1\le i<j\le n}\gamma_i\gamma_j$$

$$\le 2\left|\sum_{1\le i<j\le n}\gamma_i\gamma_j\right| \le 2\sum_{1\le i<j\le n}|\gamma_i||\gamma_j|$$

$$2R \le 2\sum_{1\le i<j\le n}|\gamma_i||\gamma_j|$$

$$LE_l(G)^2 = \left(\sum_{i=1}^{n}|\gamma_i|\right)^2$$

Thus $= \sum_{i=1}^{n}|\gamma_i|^2 + 2\sum_{1\le i<j\le n}|\gamma_i||\gamma_j|$

$$= 2R + 2R = 4R$$

$$LE_l(G) \ge 2\sqrt{R}$$

Corollary 3.2.1. Let G be a binary labeled graph of order n. Then $LE_l(G) \ge 2\sqrt{n_1 a^2 + n_2 b^2 + n_3 c^2}$

Proof: From Theorem 3.2, we have $LE_l(G) \geq 2\sqrt{R}$

$$=2\sqrt{\sum_{1\leq i<j\leq n} l_{ij}^{\ 2} + \frac{1}{n}\sum_{i=1}^{n}\left(L_i - \frac{1}{n}\sum_{j=1}^{n} L_j\right)^2}$$

$$\geq 2\sqrt{\sum_{i=1}^{n} l_{ij}^{\ 2}} = 2\sqrt{n_1 a^2 + n_2 b^2 + n_3 c^2} .$$

Theorem 3.3. Let G be a labelled graph of order n. Then

$$LE_l(G) \leq \frac{1}{n}\sum_{i=1}^{n} L_i + \sqrt{(n-1)\left[2R - \left(\frac{1}{n}\sum_{i=1}^{n} L_i\right)^2\right]}$$

Proof: We have $\gamma_n = 0 - \frac{1}{n}\sum_{i=1}^{n} L_i$.

Consider the non-negative term

$$S = \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left(|\gamma_i| - |\gamma_j|\right)^2$$

$$= 2(n-1)\sum_{i=1}^{n}\gamma_i^{\ 2} - 2\left(\sum_{i=1}^{n}|\gamma_i|\right)\left(\sum_{i=1}^{n}|\gamma_j|\right)$$

$$= 2(n-1)\left[2R - \left(\frac{1}{n}\sum_{i=1}^{n} L_i\right)^2\right] - 2\left(LE_l(G) - \frac{1}{n}\sum_{i=1}^{n} L_i\right)^2 \geq 0$$

Hence the proof.

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