Bernstein-Spectral Method for Solving Time-Fractional Heat Equation with Nonlocal Conditions

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***Abstract*—** We apply the spectral method to solve the time-fractional heat equation (T-FHE) with nonlocal condition which is utilized in different engineering and bio-science applications. In order to achieve highly accurate solution of this problem, the operational matrix of fractional derivative (described in the Caputo derivative sense) of Bernstein polynomials are used. For demonstrating the validity and applicability, numerical example is presented.

***Keywords*—**Bernstein polynomials, Time-Fractional Heat Equation, Nonlocal Conditions, Operational matrices

I. INTRODUCTION

FRACTIONAL differential equations have recently been applied in various area of engineering, science, finance, applied mathematics, bio-engineering and others [1], [2], [3], [4], [5]. In this paper, we consider the T-FHE with the nonlocal condition:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}-\frac{\partial^2 u(x,t)}{\partial x^2}=q(x,t),\ 0<x<1,\ 0<t<T,\ 0<\alpha\le 1 \quad (1)$$

$$u(x,0)=u(x,1)+f(x),\ 0<x\le 1. \quad (2)$$

$$u(0,t)=g_0(t),\ u(1,t)=g_1(t),\ 0<t\le 1. \quad (3)$$

where α is a parameter describing the fractional derivative $0<\alpha\le 1$, f, g_1, g_2 are known functions, and the function u is unknown.

We give some basic definitions and properties of the fractional calculus theory. Caputo definition of the fractional-order derivative is defined as

Definition 1.1

$$D^\alpha f(x)=\frac{1}{\Gamma(n-\alpha)}\int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha+1-n}}d(t),\ \text{n-1<α≤n, n∈N}$$

where α is the order of the derivative and n is the smallest integer greater than α. For the Caputo derivative we have:

$$D^\alpha C=0,\qquad C \text{ is a constant,} \quad (4)$$

where

$$D^\alpha x^\beta=\begin{cases}0, & \text{for } \beta\in N_0 \text{ and } \beta<\lceil\alpha\rceil,\\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}x^{\beta-\alpha}, & \text{for } \beta\in N_0 \text{ and } \beta\ge\lceil\alpha\rceil.\end{cases} \quad (5)$$

We use the ceiling function $\lceil\alpha\rceil$ to denote the smallest integer greater than or equal to α and $N_0=\{0,1,2,...\}$. Recall that for $\alpha\in N$, the Caputo differential operator coincides with the usual differential operator of integer order. The article is organized as follows: In Section 2, we summarize the properties of Bernstein and shifted Legendre polynomials, and also apply the Bernstein operational matrix of fractional derivative in section 3. Section 4 is devoted to applying the Bernstein operational matrix of fractional derivative for solving time-fractional heat equation (T-FHE) with nonlocal condition. In Section 5,the our method is applied to one example. Also a conclusion is given in Section 6.

II. LEGENDRE AND BERNSTEIN BASIS

Bernstein polynomials of degree m, on the interval $[0,1]$ as basis functions for the linear space of polynomials are defined as

$$B_{i,m}=\sum_{k=0}^{m-i}(-1)^k\binom{m}{i}\binom{m-i}{k}x^{i+k},\quad i=0,1,...,m. \quad (6)$$

A polynomial $P_m(x)$ of degree m can be expressed as

$$P_m(x)=\sum_{i=0}^{m}c_iB_{i,m}=C^T\phi(x), \quad (7)$$

where transpose of the Bernstein coefficient vector C^T and the Bernstein vector $\phi(x)$ are given by

$$C^T=[c_0,c_1,...,c_m], \quad (8)$$

and

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$$\phi(x) = [B_{0,m}(x), B_{1,m}(x), \ldots, B_{m,m}(x)]^T. \tag{9}$$

The product of Bernstein polynomials is

$$B_{i,m}(x)B_{i,n}(x) = \frac{\binom{m}{i}\binom{m}{j}}{\binom{m+n}{i+j}} B_{i+j,m+n}(x), \tag{10}$$

and

$$\int_0^1 B_{i,m}(x)dx = \frac{1}{m+1}. \tag{11}$$

Theorem 2.1 Let $\phi(x)$ be Bernstein polynomial then

$$\frac{d\phi(x)}{dx} = D_b^{(1)}\phi(x), \tag{12}$$

where $D_b^{(1)}$ is the $(m+1)\times(m+1)$ operational matrix of derivative given by

$$D_b^{(1)} = A\Lambda V, \tag{13}$$

such that A is a $(m+1)\times(m+1)$ upper triangular matrix where

$$A_{i+1,j+1} = \begin{cases} 0, & \text{for} \quad i > j \\ (-1)^{j-i}\binom{m}{i}\binom{m-i}{j-i}, & \text{for} \quad i \le j \end{cases} \tag{14}$$

$i, j = 0,1,\ldots,m$, Λ is $(m+1)\times(m)$ matrix as follows

$$\Lambda_{i+1,j+1} = \begin{cases} j, & \text{for} \quad i = j+1, \\ 0, & \text{for} \quad \text{otherwise}, \end{cases} \tag{15}$$

$i = 0,\ldots,m$, $j = 0,\ldots,m-1$. And V is $(m)\times(m+1)$ matrix can be expressed by

$$V_{k+1} = A_{k+1}^{-1}, \quad k = 0,1,\ldots,m-1, \tag{16}$$

where A_{k+1}^{-1} is $(k+1)$ th row of A^{-1}.

It is clear that:

$$\frac{d^n\phi(x)}{dx^n} = (D_b^{(1)})^n\phi(x), \tag{17}$$

where $n \in N$ and the superscript, in $D_b^{(1)}$, denotes matrix powers. Thus

$$D_b^{(n)} = (D_b^{(1)})^n, \quad n = 1,2,\ldots. \tag{18}$$

The Legendre polynomials constitute an orthonormal basis on the interval $[0,1]$, we define as follows

$$L_0(x) = 1, \qquad L_1(x) = 2x-1,$$

$$L_{i+1}(x) = \frac{(2i+1)(2x-1)}{i+1}L_i(x) - \frac{i}{i+1}L_{i-1}(x), \quad i = 1,2,\ldots.$$

The analytic form of the shifted Legendre polynomial $L_i(x)$ of degree i given by

$$L_i(x) = \sum_{k=0}^{i}(-1)^{(i+k)}\frac{(i+k)!x^k}{(i-k)!k!^2}. \tag{19}$$

Note that $L_i(0) = (-1)^i$ and $L_i(1) = 1$. The orthogonality condition is

$$\int_0^1 L_i(x)L_j(x)dx = \begin{cases} 0, & \text{for } i \ne j, \\ \frac{1}{2i+1}, & \text{for } i = j. \end{cases} \tag{20}$$

A polynomial $P_m(x)$ of degree m can be expressed as

$$P_m(x) = \sum_{j=0}^{m} l_j L_j(x) = l^T\varphi(x),$$

where the shifted Legendre coefficient vector l and the shifted Legendre vector $\varphi(x)$ are given by

$$l^T = [l_0, \ldots, l_m],$$

$$\varphi(x) = [L_0(x), L_1(x), \ldots, L_m(x)]^T. \tag{21}$$

The derivative of the vector $\varphi(x)$ can be expressed by

$$\frac{d\varphi(x)}{dx} = D_l\varphi(x),$$

where matrix D_l is the $(m+1)\times(m+1)$ operational matrix of derivative of the shifted Legendre polynomials on the interval [0,1] given by

$$D_l = d_{i+1,j+1} = \begin{cases} 2(2j+1), & \text{for} \quad j = i-k, \quad \begin{cases} k = 1,3,\ldots,m, & \text{if } m \text{ odd}, \\ k = 1,3,\ldots,m-1, & \text{if } m \text{ even}, \end{cases} \\ 0, & \text{for} \quad \text{otherwise}, \end{cases} \tag{22}$$

Lemma 2.2 Let $L_i(x)$ be shifted Legendre vector and $\alpha > 0$ then $D_l^{(\alpha)}$ is the $(m+1)\times(m+1)$ operational matrix of fractional derivative of order α in Caputo sense and is defined as follows

$$D_l^{(\alpha)}L_i(x); [\sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil}\Theta_{\lceil\alpha\rceil,0,k}, \sum_{k=\lceil\alpha\rceil}^{i}\Theta_{i,1,k}, \ldots, \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil}\Theta_{\lceil\alpha\rceil,m,k}]\varphi(x), \quad i = \lceil\alpha\rceil, \ldots, m, \tag{23}$$

where $\Theta_{i,j,k}$ is given by

$$\Theta_{i,j,k} = 2j+1\sum_{l=0}^{j}\frac{(-1)^{(i+j+k+l)}(i+k)!(l+j)!}{(i-k)!k!\Gamma(k-\alpha+1)(j-l)!(l!)^2(k+l+1-\alpha)}.$$

Note that in $D_l^{(\alpha)}$, the first $\lceil\alpha\rceil$ rows, are all zero.

III. OPERATIONAL MATRIX OF DIFFERENTIATION

By considering Legendre basis vector we have:

$$\phi(x) = W\varphi(x), \tag{24}$$

and

$$\varphi(x) = G\phi(x). \tag{25}$$

Theorem 3.1 Let D_b^α is the $(m+1)\times(m+1)$ operational matrix of fractional derivatives for any $(m+1)$-dimensional fractional Bernstein functions vector $\phi(x)$ we have $\frac{d^\alpha}{dx^\alpha}\phi(x) = D_b^\alpha \phi(x)$, the D_b^α can be obtained as follow:

$$D_b^\alpha = W D_l^\alpha G, \tag{26}$$

where, W and G are $(m+1)\times(m+1)$ transforming matrices, and also D_l^α is a $(m+1)\times(m+1)$ matrix of operational matrix of fractional derivative of the shifted Legendre polynomials [5].

IV. SOLUTION OF THE PROBLEM

Suppose $\phi(x)$ and $\phi(t)$ are vectors of Bernstein polynomials on [0,1]. Now the unknown function $u(x,t)$ in Eq. (1) can be approximated as

$$u(x,t); \phi^T(x)U\phi(t), \tag{27}$$

Also, we have

$$u_{xx}(x,t) = (D_b^2\phi(x))^T U\phi(t) = \phi^T(x)(D_b^2)^T U\phi(t), \tag{28}$$

So, we obtain:

$$\phi^T(x)UD_b^\alpha\phi(t) = f(x,t)\phi^T(x)(D_b^2)^T U\phi(t), \tag{29}$$

we now collocate Eq. (29) in $(m-1)\times(m)$ points (x_i, t_j), $i = 2,...,m,\ j = 2,...,m+1,$

$$R(x_i, t_j) = \varphi^T(x_i)UD_b^\alpha\varphi(t_j) -$$

$$f(x,t)\varphi^T(x_i)(D_b^2)^T U\varphi(t_j)) - q(x,t) = 0, \tag{30}$$

$$i = 2,...,m,\ j = 2,...,m+1.$$

where x_i and t_j, are shifted points of L_j. Collocating Eqs. (2) and (3) in $m+1$ points x_i, $i = 1,...,m+1,$ and m points t_j, $j = 1,...,m,$ we gain

$$u(0,t_j) = g_0(t_j),\quad u(1,t_j) = g_1(t_j) \quad j = 1,2,...,m, \tag{31}$$

$$u(x_i,0) = u(x_i,1) + f(x_i),\quad i = 1,...,m+1. \tag{32}$$

Hence we solve generated system.

V. NUMERICAL RESULTS

Example 5.1 We consider Eqs. (1)- (3) with $1 < \alpha \le 2$ and the given data:

$$f(x,t) = -sin(2\pi x),\quad g_1(t) = 0,\quad g_2(t) = 0,$$

$$q(x,t) = \frac{2t^{1.5}sin(2\pi x)}{\Gamma(1.5)} + 4(\pi)^2 t^2 sin(2\pi x)$$

The exact solution is: $t^2 sin(2\pi x)$.

TABLE I

ABSOLUTE VALUES OF ERROR FOR $u(0.5,t)$ FROM EXAMPLE 1

x	$t=0.1$	$t=0.2$	$t=0.3$
0.5	3.709×10^{-9}	2.969×10^{-9}	3.24310^{-9}
	$t=0.4$	$t=0.5$	$t=0.6$
	3.918×10^{-9}	5.040×10^{-9}	6.484×10^{-9}
	$t=0.7$	$t=0.8$	$t=0.9$
	8.306×10^{-9}	1.0448×10^{-9}	1.2913×10^{-8}
	$t=1$		
	1.0448×10^{-9}		

VI. CONCLUSION

The properties of the Bernstein polynomials are used to solve time-fractional heat equation (T-FHE) with nonlocal condition by reducing the problem to the system of equations. From the solutions obtained using the suggested method we can conclude that these solutions are in excellent agreement with the exact solution.

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