Functionally Graded Axisymmetric Rotating Annular Disc with Internal and External Pressure and Constant Poisson's Ratio

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***Abstract*—** A closed form solution is obtained for an axisymmetric functionally graded rotating annular disc using stress function. Stress analysis under variation in Young's modulus and density is studied with and without the effect of internal and external pressure. Scientists have solved problem on functionally graded materials with the variation in profile of the disc. In this paper, Poisson ratio is taken as constant and the Young's modulus and density profile depends on the radial coordinate only.

The purpose of this research is to find an analytical solution of a thin circular annular rotating disc having a variable density and modulus profile. The pressure is applied at the internal and external surface of the disc and the results are compared with those available in literature without pressure. The results obtained are discussed numerically and depicted graphically.

***Keywords*—** Annular disc, Axisymmetric, FGM, Pressure

I. Introduction

IN the scientific community the most popular topic is functionally graded materials. In this, the study of the variation in material properties likes Young's modulus, Poisson ratio; density, thickness etc. are region of interest to the researchers. The variations observed in FGM are smooth and continuous in consecutive surface of different materials. It can be observed that the properties of FGM are different from that of its constituents.

Failure of aircraft and aerospace missiles, breakdown of various machines due to over-heating and other properties which could not resist barriers, such problems leads to introduction to FGM. In FGM materials having different properties having huge differences in their M.P and B.P., bring across to prepare a new material to deal with such situations. Now-a-days, we have wide use of FGM in rocket heat shields, heat exchanger tubes, wear-resistant linings etc. In aircraft and aerospace industries and computed circuit industries are very much interested in such materials which can withstand very high temperature gradients. Basically, FGMs are generated to use most of the material properties in the best possible ways.

The research on FGM started way back in 1984 by Koizumi [1] in Japan during the space plane project. Since then several researchers have published their papers in many internationally reputed journals related to FGM and has shown its application to various fields. Reddy et al. [2] developed a finite element model for solving the problem related to functionally graded axisymmetric cylinder under thermal barrier. In 1999 [3], the authors have studied the bending and stretching of solid and annular circular plates using the first – order shear deformation Mindlin plate theory. In 2003 Chakraborty et al. [4] has extended the work of Reddy in which new method is developed to study the thermo elastic behavior of functionally graded beam structures. A review paper on FGM is published by Birman et al. [5] in 2007. Alavi et al. [6] has solved problem on spherical shells made of FGM under thermal gradients and mechanical loads. Creep behavior of rotating disc made of isotropic material is studied for FGM by Rattan et al. [7].

In 2011, Carrera et al. [8] studied the behavior of thickness variation on functionally graded plates and shells. The effect of non-homogeneity on the response of linearly elastic isotropic hollow circular cylinders or disks uniform internal or external pressure was investigated by Horgan et al. [9]. With the continuation of the previous work, Callioglu [10] has done stress analysis of functionally graded disc under temperature distribution and external load. Callioglu et al. [11] has analyzed the properties of disc under various temperature distributions and internal pressure. Analytical and numerical methods are suggested for solving problem related to rotating disc by Callioglu et al. [12] in which FEM is used for numerical method. Stress and strain analysis for a disc under constant rotation is done by Sharma et al. [13] under temperature distribution using numerical method. Rosyid et al. [14] had taken a problem of an inhomogeneous rotating disc with the thickness varying arbitrarily in the radial direction and solved the problem using finite element method. Sahni et al. [15] taken the approach of Callioglu solved the problem of FGM under internal pressure.

In this paper, the elasticity modulus (E) and density (ρ) of the disc are a uniform function of radius 'r', so that these functions are called functionally graded materials. Poisson's ratio is assumed to be constant. For functionally graded annular rotating discs at constant angular speed under internal and external pressure, closed-form solutions using the infinitesimal theory of elasticity are obtained. Results obtained are depicted graphically and compared with the known solution without variation in density and Young's modulus by

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Callioglu [12].

II. Problem Description

Consider an annular disc of thin – walled with internal and external radii 'a' and 'b' respectively. It is subjected to internal and external pressure $p1$ and $p2$ respectively and rotating with a constant angular speed ω.

The *equation of continuity* to be satisfied is given below,

$$\frac{d}{dr}\{rT_{rr}\}-T_{\theta\theta}+\rho(r)\omega^2r^2=0 \quad (1)$$

where $\rho(r)$ is the density of the material varying with position vector 'r'.

In this research paper, the problem considered is of functionally graded materials under plane stress with the variation of Young's modulus and density in a form, given as

$$E(r)=E_0r^n, \quad \rho(r)=\rho_0(1+n*(r+r^2)) \quad (2)$$

Here E_0 and ρ_0 are Young's modulus and density at $n=0$ respectively.

Defining a function of stress satisfying the equation of continuity (1) as

$$T_{rr}=\frac{F}{r} \text{ and } T_{\theta\theta}=\frac{dF}{dr}+\rho(r)\omega^2r^2 \quad (3)$$

The relation between the strains – radial and circumferential with the radial displacement due to rotational symmetry are given as

$$\varepsilon_r=\frac{du}{dr}, \quad \varepsilon_\theta=\frac{u}{r} \text{ and } \upsilon_{r\theta}=0 \quad (4)$$

where ε_r and ε_θ are strains along radial and circumferential direction respectively and $\upsilon_{r\theta}$ is the shearing strain which is zero. Here *u* is the displacement in the radial direction.

The *strain compatibility equation* derived from equation is given as,

$$\varepsilon_r=\varepsilon_\theta+r\frac{d}{dr}(\varepsilon_\theta) \quad (5)$$

The stress - strain relationship is defined as

$$\varepsilon_r=\frac{1}{E(r)}(T_{rr}-\upsilon T_{\theta\theta}) \quad \text{and} \quad \varepsilon_\theta=\frac{1}{E(r)}(T_{\theta\theta}-\upsilon T_{rr}) \quad (6)$$

Now using (2), (3) and (6) into (5), we get

$$r^2F''-rF'-(1-n\nu)\mathrm{F}=r^2\left(-3+n\big(1+r*n*(1+r)\big)-\nu-\mathrm{r}*\mathrm{n}*\big(4+\nu+\mathrm{r}(5+\nu)\big)\right) \quad (7)$$

The stress function F is calculated as

$$F=C1*r^{\frac{1}{2}\left(n-\sqrt{4+n^2-4n\nu}\right)}+C2*r^{\frac{1}{2}\left(n+\sqrt{4+n^2-4n\nu}\right)}+\frac{n*r^5*(-5+n-\nu)\rho_0\omega^2}{24+n(-5+\nu)}+\frac{n*r^4*(-4+n-\nu)\rho_0\omega^2}{15+n(-4+\nu)}+\frac{r^3*(-3+n-\nu)\rho_0\omega^2}{8+n(-3+\nu)} \quad (8)$$

The radial and circumferential stresses are calculated from (3) by substituting (8).

The boundary conditions are defined as

$$T_{rr}=-p1 \quad \text{at} \quad r=a$$
$$T_{rr}=-p2 \quad \text{at} \quad r=b \quad (9)$$

Using boundary conditions (9) in (8), we get the integration constants C1 and C2 as

$$\mathrm{C1}=(a^{\frac{1}{2}(-n+\sqrt{4+n^2-4n\nu})}b^{\frac{1}{2}(-n+\sqrt{4+n^2-4n\nu})}(-a^{\frac{1}{2}(n+\sqrt{4+n^2-4n\nu})}\mathrm{p2}(24+n(-5+\nu))(15+n(-4+\nu))(8+n(-3+\nu)) - a^{\frac{1}{2}(n+\sqrt{4+n^2-4n\nu})}b^5n(15+n(-4+\nu))(8+n(-3+\nu))(-5+n-\nu)\rho0\omega^2 - a^{\frac{1}{2}(n+\sqrt{4+n^2-4n\nu})}b^4n(24+n(-5+\nu))(8+n(-3+\nu))(-4+n-\nu)\rho0\omega^2 - a^{\frac{1}{2}(n+\sqrt{4+n^2-4n\nu})}b^3(24+n(-5+\nu))(15+n(-4+\nu))(-3+n-\nu)\rho0\omega^2 + b^{\frac{1}{2}(n+\sqrt{4+n^2-4n\nu})}(\mathrm{p1}(24+n(-5+\nu))(15+n(-4+\nu))(8+n(-3+\nu)) + a^3(-360(3+\nu) + n(873+a^2(15+n(-4+\nu))(8+n(-3+\nu))(-5+n-\nu) + a(24+n(-5+\nu))(8+n(-3+\nu))(-4+n-\nu) + n^2(-5+\nu)(-4+\nu) + 54\nu - 39\nu^2 - n(231+\nu(-46+(-6+\nu)\nu))))\rho0\omega^2)))/((a^{\sqrt{4+n^2-4n\nu}} - b^{\sqrt{4+n^2-4n\nu}})(24+n(-5+\nu))(15+n(-4+\nu))(8+n(-3+\nu)))$$

and

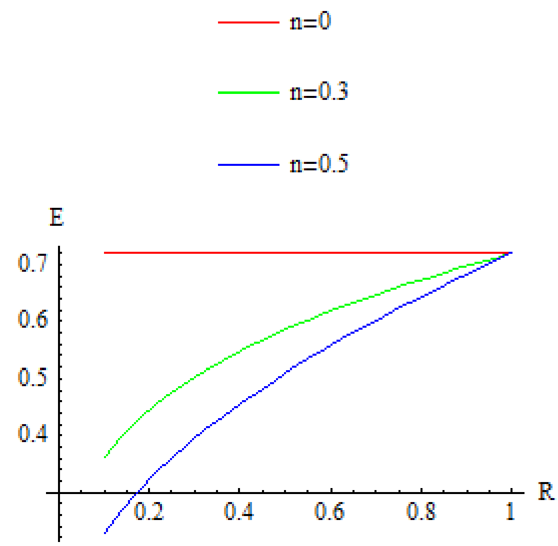
$$\mathrm{C2}=(a^{-n/2}b^{-n/2}(-a^{\frac{1}{2}\sqrt{4+n^2-4n\nu}}b^{n/2}\mathrm{p1}(24+n(-5+\nu))(15+n(-4+\nu))(8+n(-3+\nu)) - a^{5+\frac{1}{2}\sqrt{4+n^2-4n\nu}}b^{n/2}n(15+n(-4+\nu))(8+n(-3+\nu))(-5+n-\nu)\rho0\omega^2 - a^{4+\frac{1}{2}\sqrt{4+n^2-4n\nu}}b^{n/2}n(24+n(-5+\nu))(8+n(-3+\nu))(-4+n-\nu)\rho0\omega^2 - a^{3+\frac{1}{2}\sqrt{4+n^2-4n\nu}}b^{n/2}(24+n(-5+\nu))(15+n(-4+\nu))(-3+n-\nu)\rho0\omega^2 + a^{n/2}b^{\frac{1}{2}\sqrt{4+n^2-4n\nu}}(\mathrm{p2}(24+n(-5+\nu))(15+n(-4+\nu))(8+n(-3+\nu)) + b^3(-360(3+\nu) + n(873+b^2(15+n(-4+\nu))(8+n(-3+\nu))(-5+n-\nu) + b(24+n(-5+\nu))(8+n(-3+\nu))(-4+n-\nu) + n^2(-5+\nu)(-4+\nu) + 54\nu - 39\nu^2 - n(231+\nu(-46+(-6+\nu)\nu))))\rho0\omega^2)))/((a^{\sqrt{4+n^2-4n\nu}} - b^{\sqrt{4+n^2-4n\nu}})(24+n(-5+\nu))(15+n(-4+\nu))(8+n(-3+\nu)))$$

III. Numerical Discussion and Illustration

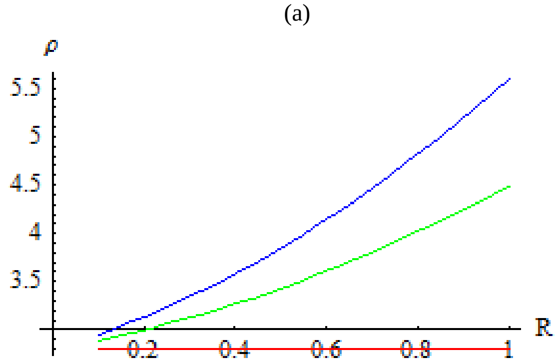
In this paper, stresses – radial and circumferential are calculated by varying geometric parameter n and different values of internal and external pressure. The values that are being taken are listed below:

a = 40 mm, b = 100 mm, $\omega=250\ rad/s$, $E_0=72000$ MPa, $\rho_0=2.8$, $\nu=0.3$, n = 0, 0.3, 0.5, p1 = 0, 5, 10, 20, p2 = 0, 5, 10, 20.

In Fig. 1, Young's modulus and density variations are seen for different values of their geometric parameter 'n' respectively. It is seen that the modulus and density remains constant when n = 0. With the increase in value of 'n', Young's modulus (E) decreases and density (ρ) increases as shown in the Fig. 1. With the increase in radii ratio (r/b = R) both increases except for n = 0.



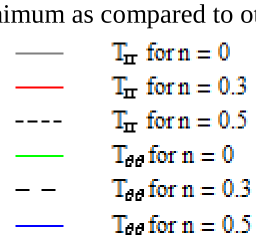
(a)

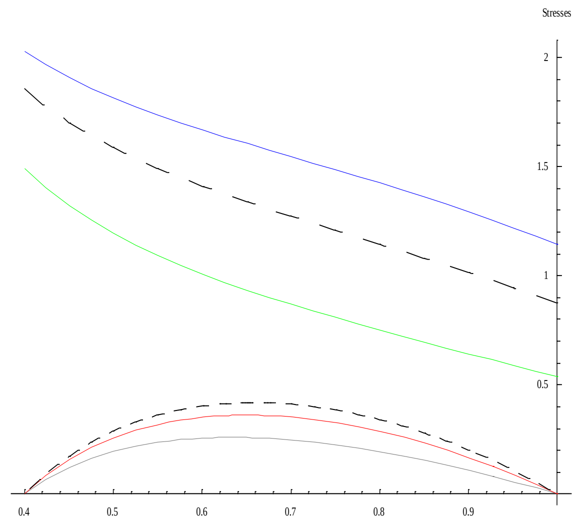


(b)

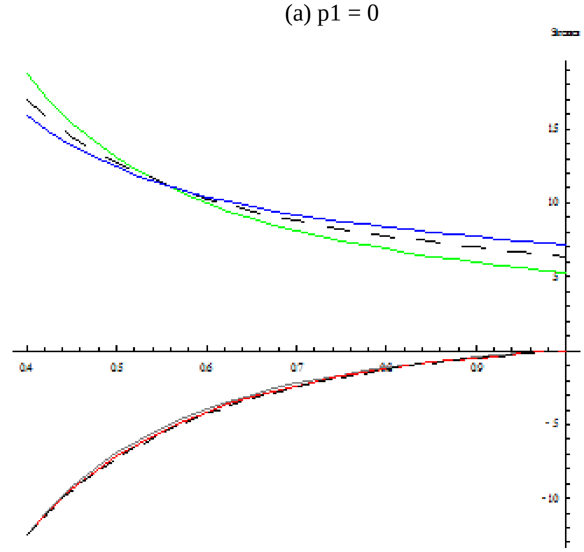
Fig. 1: Young's modulus (a) and density (b) variation calculated for different values of geometric parameter n against different radii ratios.

From Fig. 2(a), when both internal and external pressures are zero, the radial and circumferential stresses increase with the increase in geometric parameter (n). The stresses – radial and circumferential are tensile in the absence of pressure. The circumferential stress is maximum at the internal radii and minimum at the external radii, whereas the radial stress is maximum in between internal and external. With the increase in internal pressure as in Fig. 2(b, c, d) and external pressure zero the radial stresses become compressive and circumferential stress remains tensile. With the increase in internal pressure the circumferential stress increases significantly, to counter the effect of radial stresses. From Fig. 2(b), it is observed that for n = 0, the circumferential stress is maximum at the internal radii but as we proceed towards the outer radii, it is minimum as compared to other values of n.

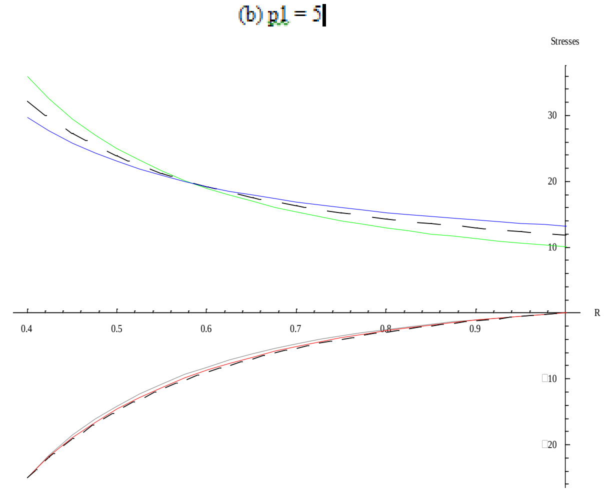




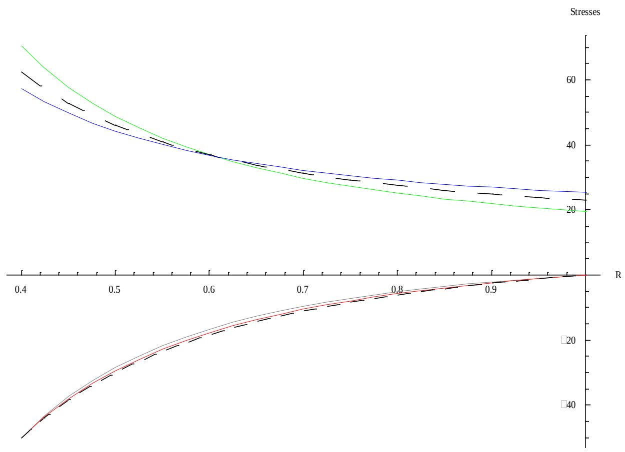
(a) p1 = 0



(b) p1 = 5



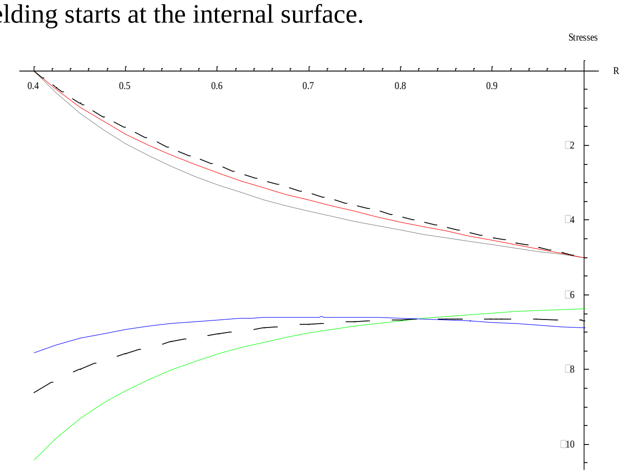
(c) p1 = 10



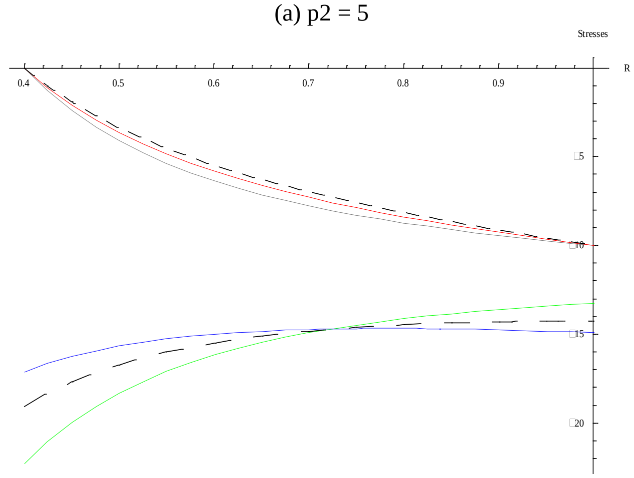
(d) p1 = 20

Fig. 2: Circumferential and radial stresses under various values of internal pressure.

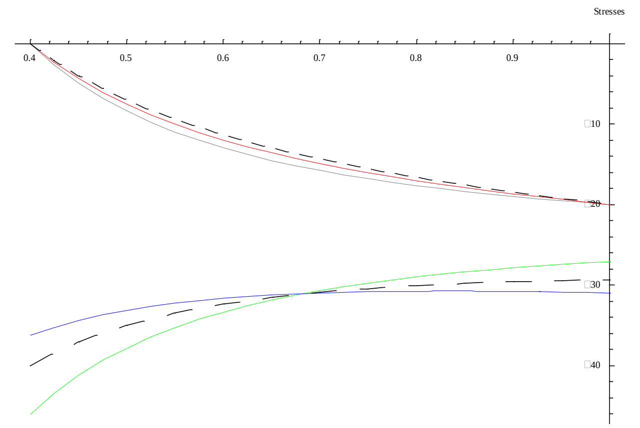
In Fig. 3, the stresses are calculated under external pressure. Under external pressure, the radial and circumferential stress increases with the increase in radii ratio and are compressive. The difference is maximum at the internal surface; hence the yielding starts at the internal surface.



(a) p2 = 5



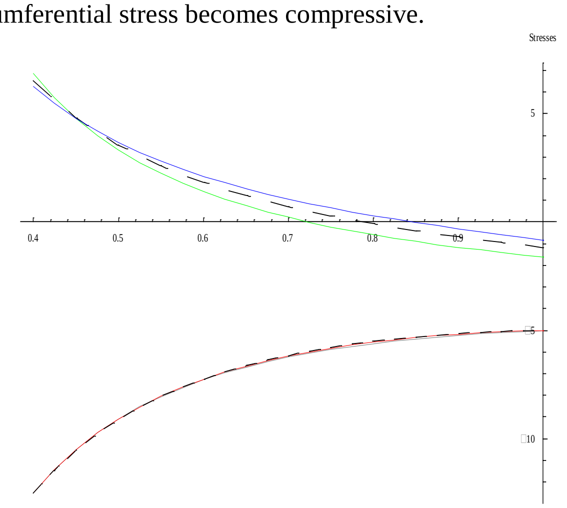
(b) p2 = 10



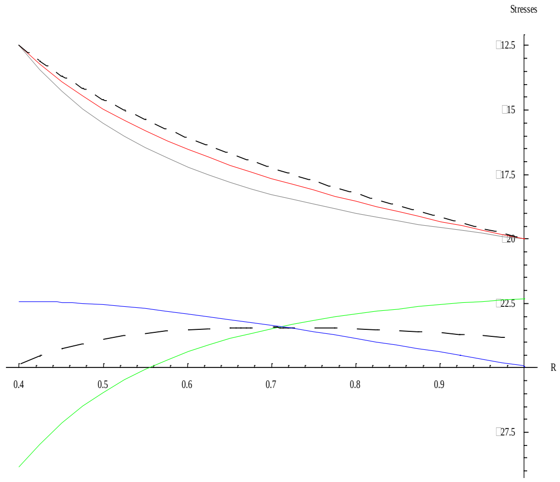
(c) p2 = 20

Fig. 3: Radial and Circumferential stresses for various values of external pressure.

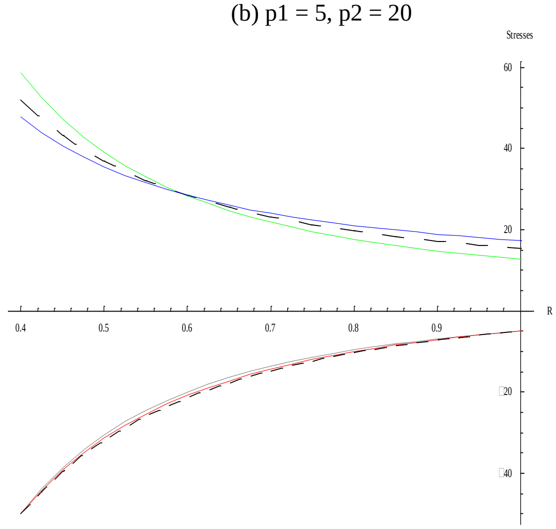
From Fig. 4, the graphs are drawn for various values of internal and external pressure against radii ratio. Keeping the internal pressure constant (Fig. 4 (a) and (b)) and increasing the external pressure, the circumferential stress becomes more compressive. The circumferential stress is maximum at the internal surface for n = 0, 0.3 and is maximum at the external surface for n = 0.5. From Fig. 4 (a) and (c) when the external pressure is constant, and internal increases the circumferential stress shows a sharp increase. From Fig. 4(c), it is seen that for n = 0, the circumferential stress is maximum at the internal radii and minimum at the external radii as compared to other geometric values. From Fig. 4 (a) and (d) the circumferential stress is maximum at the internal radii but at R = 0.68, the circumferential stress becomes compressive.



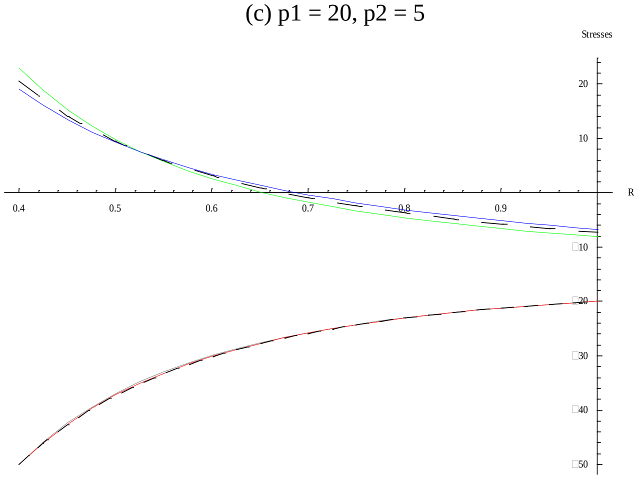
(a) p1 = 5, p2 = 5



(b) p1 = 5, p2 = 20



(c) p1 = 20, p2 = 5



(d) p1 = 20, p2 = 20

Fig. 4: Radial and Circumferential Stresses under variation of internal and external pressure.

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