Application of a Generalized Differential Operator to Analytic Functions Related to Conic Domains

Ajab B. Akbarally, and Ahmad R. Azudin

Abstract—We introduce a new class \( S_{P}^{m,a} (\beta, \gamma) \) of analytic functions by applying a generalized differential operator \( LD_{P}^{m,a} \). The object of this paper is to determine the coefficient estimates, sufficient condition, growth and distortion theorems for functions \( f(z) \) belonging to the class \( S_{P}^{m,a} (\beta, \gamma) \).

Keywords—Analytic functions, differential operator, coefficient estimates, sufficient condition, growth and distortion theorems.

I. INTRODUCTION

Let \( A \) denote the class of functions of the form
\[
f(z) = z + a_2 z^2 + a_3 z^3 + \ldots = z + \sum_{k=2}^{\infty} a_k z^k
\]
which are analytic, univalent in the open unit disk \( U = \{ z: z \in C, |z| < 1 \} \) and normalized by the conditions \( f(0) = 0 \) and \( f'(0) = 1 \).

Let \( S^* \) denote the class of starlike functions. Analytically, a function \( f \in A \) is said to be a starlike function if and only if
\[
\text{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} \geq 0 \quad \text{for} \quad z \in U.
\]
Let \( K \) denote the class of convex functions. Analytically, a function \( f \in A \) is said to be a convex function if and only if
\[
\text{Re}\left\{ 1 + \frac{zf'(z)}{f'(z)} \right\} \geq 0 \quad \text{for} \quad z \in U.
\]

A function \( f(z) \) is uniformly starlike (or uniformly convex) in \( U \) if \( f(z) \) is in \( S^* (K) \) and has the property that for every circular arc \( \zeta \) contained in \( U \), with center \( \xi \) also in \( U \), the arc \( f(\zeta) \) is starlike (convex) with respect to \( f(\xi) \). The class of uniformly starlike functions is denoted by \( S_p \) and the class of uniformly convex functions by \( UCV \) (Goodman,1991a,1991b).

Ronning (1993) and Ma and Minda (1994) gave the following conditions:
\[
f \in S_p \quad \Leftrightarrow \quad \text{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} \geq \left| \frac{zf'(z)}{f(z)} - 1 \right|
\]
and
\[
f \in UCV \quad \Leftrightarrow \quad \text{Re}\left\{ 1 + \frac{zf'(z)}{f'(z)} \right\} \geq \left| \frac{zf'(z)}{f'(z)} \right|.\]

Note that \( f(z) \in UCV \Leftrightarrow zf'(z)/f'(z) \in S_p \).

Bharati, Parvatham and Swaminathan (1997) defined \( \beta - S_p(\gamma) \) and \( \beta - UCV(\gamma) \) as the classes of \( \beta \)-uniformly starlike functions and \( \beta \)-uniformly convex functions of order \( \gamma \) where functions in these classes satisfy the condition
\[
\text{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} \geq \beta \left| \frac{zf'(z)}{f(z)} - 1 \right| + \gamma
\]
and
\[
\text{Re}\left\{ 1 + \frac{zf'(z)}{f'(z)} \right\} \geq \beta \left| \frac{zf'(z)}{f'(z)} \right| + \gamma \quad \text{with} \quad 0 \leq \beta < \infty \quad \text{and} \quad 0 \leq \gamma < 1 \quad \text{respectively}.
\]

The geometric interpretation is as given below (Al-Oboudi and Al-Amoudi (2008)):
\[
f \in \beta - UCV(\gamma) \quad \text{and} \quad \beta - S_p(\gamma) \quad \text{if and only if} \quad 1 + \frac{zf'(z)}{f'(z)} \quad \text{and} \quad \frac{zf'(z)}{f'(z)}
\]
respectively, take all the values in the conic domain \( R_{\beta,\gamma} \) which is included in the right half plane such that
\[
R_{\beta,\gamma} = \left\{ u + iv: u > \beta \sqrt{(u-1)^2 + v^2} + \gamma \right\}.
\]

Denote by \( P(\beta,\gamma) \), \((\beta \geq 0, 0 \leq \gamma < 1)\) the family of functions \( p \), such that \( p \in P \), where \( P \) denotes the well-known class of Caratheodory functions and \( p \prec P_{\beta,\gamma} \) in \( U \). (An analytic function \( f \) in \( U \) is said to be subordinate to an analytic function \( g \) in \( U \) (denoted by \( f \prec g \)) if \( f(z) = g(w(z)) \), \( z \in U \) for some analytic function \( w(z) \) with \( w(0) = 0 \) and \( |w(z)| < 1, \ z \in U \).

The function \( P_{\beta,\gamma} \) maps the unit disk conformally onto the domain \( R_{\beta,\gamma} \) such that \( 1 \in R_{\beta,\gamma} \) and \( \partial R_{\beta,\gamma} \) is a curve defined by the equality

Ajab B. Akbarally, Department of Mathematics, Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA,40450 Shah Alam, Selangor, Malaysia. (Email ID – ajab@tmsk.uitm.edu.my)

Ahmad R. Azudin, Department of Mathematics, Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA,40450 Shah Alam, Selangor, Malaysia. (Email ID – adi_kisas@yahoo.com)

http://dx.doi.org/10.15242/IIE.E0315028
\[ \partial R_{\beta,\gamma} = \left\{ u + iv : u^2 > \left( \beta \sqrt{(u-1)^2 + v^2} + \gamma \right)^2 \right\}. \]

From elementary computations we see that \( \partial R_{\beta,\gamma} \) represents conic sections symmetric about the real axis. Thus, \( R_{\beta,\alpha} \) is an elliptic domain for \( \beta > 1 \), a parabolic domain for \( \beta = 1 \), a hyperbolic domain for \( 0 < \beta < 1 \) and a right half plane \( u > \gamma \) for \( \beta = 0 \).

Aghalary and Azadi (2005) obtained the functions \( P_{\beta,\gamma} \) which play the role of extremal functions of \( P(P_{\beta,\gamma}) \) as

\[
P_{\beta,\gamma}(z) = \begin{cases}
\frac{1 + (1 + 2\gamma)u}{1 - z} & ; \beta = 0 \\
1 + \frac{2(1 - \gamma)}{\pi^2} \left( \frac{1 + \sqrt{1 - z^2}}{1 - z} \right) - 1 - \beta^2 & ; \beta > 1 \\
\frac{(1 - \gamma) \cos \left( \frac{2}{\pi} \cos^{-1} \beta \right)}{1 - \beta^2} - \frac{\beta - \gamma}{1 - \beta^2} & ; 0 < \beta < 1 \\
\frac{(1 - \gamma) \left( \frac{1}{2\pi(t)} \right) u(t)}{\beta^2 - 1} - \frac{1}{\sqrt{1 - x^2} - 1 - \beta^2} & ; \beta > 1
\end{cases}
\]

where \( u(z) = z - \frac{\sqrt{1 - z^2}}{1 - z} \), \( z \in U \), \( t \in (0,1) \) is chosen such that
\[ \beta = \cos \left( \frac{\pi \kappa'(t)}{4\kappa(t)} \right) \quad , \quad \kappa(t) = \text{Legendre's complete elliptic integral of the first kind and } \kappa'(t) = \text{complementary integral of } \kappa(t) \].

Clearly, for \( \beta = 0 \),
\[ P_{0,\alpha}(z) = 1 + 2(1 - \gamma)z + 2(1 - \gamma)z^2 + 2(1 - \gamma)z^3 + \ldots \]

For \( \beta = 1 \), from Ronning (1993) and Ma and Minda (1992) we have that
\[ P_{1,\alpha}(z) = 1 + \frac{8}{\pi^2} (1 - \gamma)z + \frac{16}{3\pi^2} (1 - \gamma)z^2 + \ldots \]

Also by comparing Taylor series expansions by Kanais and Yaguchi (2001), they obtained for \( 0 < \beta < 1 \), where
\[ B = \frac{2}{\pi} \sin^{-1} \beta, \]
\[ P_{\beta,\gamma}(z) = 1 + \frac{(1 - \gamma)}{1 - \beta^2} + \sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{B(n-1)\gamma}{(2\ell)(2n-1)} \left( \frac{B(n-1)\gamma}{(2\ell)(2n-1)} \right)^n z^n, \]

and for \( \beta > 1 \),
\[ P_{\beta,\gamma}(z) = 1 + \frac{(1 - \gamma)}{4\sqrt{1 - k^2}(1 + t)} \left( \frac{1}{2\sqrt{1 - k^2}(1 + t)} \right)^{2n} \left( 1 - \frac{4}{k^2} (k^2 + 6 + k + \pi^2 z^2) z^2 + \ldots \right). \]

Ruscheweyh (1975) introduced the differential operator
\[ R^m : A \to A, \quad m \in N_0 = \{0,1,2,\ldots\} \]
\[ R^m f(z) = z + \sum_{k=2}^{\infty} \frac{G(k+m)(k-1)!G(1+m)}{G(k-1)!G(1+m)} a_k z^k. \]

Salagean (1983) defined \( S^m : A \to A, \quad m \in N_0 = \{0,1,2,\ldots\} \) as
\[ S^m f(z) = z + \sum_{k=2}^{\infty} k^m a_k z^k. \]

Using these two operators Lupas (2013) introduced the operator \( L^m_{\lambda} \), \( m \in N_0 = \{0,1,2,\ldots\}, \quad \lambda \geq 0 \) which is defined as
\[ L^m_{\lambda} f(z) = \frac{(1 - \lambda) R^m f(z) + \lambda S^m f(z)}{(1 - \lambda) R^m f(z) + \lambda S^m f(z)} \quad , \quad m \in N_0 = \{0,1,2,\ldots\}, \quad \lambda \geq 0. \]

Ruscheweyh and Salagean operators can be obtained by setting \( \lambda = 0 \) and \( \lambda = 1 \) respectively in (4).

Al-Oboudi and Al-Amoudi (2008) introduced a new linear multiplier fractional differential operator \( D^{m,\alpha}_{\mu} f(z) \) where
\[ m \in N_0 = \{0,1,2,\ldots\}, \quad 0 \leq \alpha < 1, \quad \lambda \geq 0, \quad \mu \geq 0. \]

It is defined as
\[ D^{m,\alpha}_{\mu} f(z) = z + \sum_{k=2}^{\infty} \Psi_{k,m}(\alpha, \mu) a_k z^k \]

where
\[ \Psi_{k,m}(\alpha, \mu) = \left[ \frac{\Gamma(k+1)\Gamma(2 - \alpha)}{\Gamma(k + 1 - \alpha)} (1 + \mu(k-1)) \right]^m. \]

By using the operator \( L^m_{\lambda} \) defined by Lupas (2013) and the operator \( D^{m,\alpha}_{\mu} \) defined by Al-Oboudi and Al-Amoudi (2008) we define a new generalized differential operator \( LD^{m,\alpha}_{\lambda,\mu} \) as
\[ LD^{m,\alpha}_{\lambda,\mu} f(z) = (1 - \lambda) R^m f(z) + \lambda D^{m,\alpha}_{\mu} f(z) \]

where \( m \in N_0 = \{0,1,\ldots\}, \quad 0 \leq \alpha < 1, \quad \lambda \geq 0, \quad \mu \geq 0. \) If \( f \) is given by (1), then we have that
\[ LD^{m,\alpha}_{\lambda,\mu} f(z) = z + \sum_{k=2}^{\infty} (1 - \lambda) C(m,k) + \lambda \Psi_{k,m}(\alpha, \mu) a_k z^k. \]

When \( \alpha = 0 \) and \( \mu = 1 \), we get Lupas operator. For \( \lambda = 1 \), we get Al-Oboudi and Al-Amoudi operator. For \( \lambda = 0 \), we get Ruscheweyh operator. By setting \( \lambda = 1 \) and \( \alpha = 0 \), we get Al-Oboudi operator (2004). When \( \lambda = 1, \alpha = 0, \mu = 1 \), we obtain Salagean operator. Owa-Srivastava operator (1987) can be obtained by considering \( \lambda = 1, m = 1, \mu = 0 \).

**Definition 1.** Let \( f \) be of the form (1), \( 0 \leq \beta < \infty, \quad m \in N_0 = \{0,1,\ldots\}, \quad 0 \leq \alpha < 1, \lambda \geq 0, \mu \geq 0. \) Then \( f \in SP^{m,\alpha}_{\lambda,\mu}(\beta, \gamma) \) if and only if \( f \) satisfies the condition
\[ \Re\left[ \frac{z(LD^{m,\alpha}_{\lambda,\mu} f(z))}{LD^{m,\alpha}_{\lambda,\mu} f(z)} \right] > \beta \frac{z(LD^{m,\alpha}_{\lambda,\mu} f(z))}{LD^{m,\alpha}_{\lambda,\mu} f(z)} - 1 + \gamma. \]

Observe that \( f \in SP^{m,\alpha}_{\lambda,\mu}(\beta, \gamma) \) if and only if \( LD^{m,\alpha}_{\lambda,\mu} f(z) \in \beta - SP(\gamma) \).

Let \( f \) be of the form (1), \( 0 \leq \beta < \infty, \quad m \in N_0 = \{0,1,\ldots\}, \quad 0 \leq \alpha < 1, \lambda \geq 0, \mu \geq 0. \) Then \( f \in UCV^{m,\alpha}_{\lambda,\mu}(\beta, \gamma) \) if and only if \( f \) satisfies the condition
\[ \Re\left[ z(LD^{m,\alpha}_{\lambda,\mu} f(z))^\ast \right] > \beta \frac{z(LD^{m,\alpha}_{\lambda,\mu} f(z))^\ast}{LD^{m,\alpha}_{\lambda,\mu} f(z)} + \gamma. \]

Observe that \( f \in UCV^{m,\alpha}_{\lambda,\mu}(\beta, \gamma) \) if and only if \( LD^{m,\alpha}_{\lambda,\mu} f(z) \in \beta - UCV(\gamma) \).
Geometric interpretation. From (7), \( f \in SP_{k,\mu}^{m,\alpha}(\beta, \gamma) \) if and only if 
\[
q(z) = \frac{z(LD_{k,\mu}^{m,\alpha} f(z))'}{LD_{k,\mu}^{m,\alpha} f(z)}
\]
takes all the values in the conic domain \( R_{\beta,\gamma} \) given in (2) which is included in the right half plane.

It is noticed that by specializing the parameters \( m, \alpha, \lambda, \mu, \beta, \) and \( \gamma \), the class \( SP_{k,\mu}^{m,\alpha}(\beta, \gamma) \) can be reduced to the following classes studied by various authors:
\[
SP_{1,\mu}^{0,\alpha}(0,0) \equiv S^* \quad \text{(see Duren (1983))} ; \\
SP_{1,\mu}^{0,\alpha}(1,0) \equiv S_p \quad \text{(see Ronning (1993))} ; \\
SP_{1,\mu}^{0,\alpha}(\beta,0) \equiv S_p \quad \text{(see Kanasa and Winiowska(2000))} ; \\
SP_{1,\mu}^{0,\alpha}(\beta,\gamma) \equiv S_p\gamma \quad \text{(see Bharati, Parvatham and Swaminathan (1997))} ; \\
SP_{1,\mu}^{0,\alpha}(\beta,\gamma) \equiv \beta - S_p\gamma \quad \text{(see Akbarally and Darus (2007))} ; \\
SP_{1,\mu}^{m,\alpha}(\beta,0) \equiv S_n \quad \text{(see Mishra and Gochayat (2008))} ; \\
SP_{1,\mu}^{m,\alpha}(\beta,\gamma) \equiv S_p\gamma \quad \text{(see Al-Oboudi and Al-Amoudi (2008))} ; \\
SP_{1,\mu}^{m,\alpha}(0,\gamma) \equiv ST^n \quad \text{(see Salagean (1983))} ; \\
SP_{1,\mu}^{m,\alpha}(\gamma,\gamma) \equiv ST\gamma \quad \text{(see Srivastava, Mishra, and Das (1988))} ; \\
SP_{1,\mu}^{m,\alpha}(0,\gamma) \equiv \beta - S_p\gamma \quad \text{(see Srivastava and Mishra (2000))} ; \\
SP_{1,\mu}^{m,0}(\beta,0) \equiv \beta - S_p^n \quad \text{(see Kanasa and Yugeuchi (2001)).}
\]

The aim of this paper is to investigate several basic properties of the class \( SP_{k,\mu}^{m,\alpha}(\beta, \gamma) \).

II. COEFFICIENT BOUNDS

Here we give bounds for the coefficients of series expansion of functions belonging to the class \( SP_{k,\mu}^{m,\alpha}(\beta, \gamma) \).

Theorem 2. Let \( (\alpha_k) \) is the Pochammer symbol defined in terms of Gamma functions by
\[
(\alpha_k) = \binom{\alpha + k}{\alpha} = \frac{\Gamma(\alpha + k)}{\Gamma(\alpha)} \quad \text{;} \quad \alpha = 0, 1, 2, \ldots.
\]
If a function \( f \) of the form (1) is in \( SP_{k,\mu}^{m,\alpha}(\beta, \gamma) \), then
\[
|\alpha_k| \leq \frac{1}{(1 - \gamma)C(m,k) + \lambda \Psi_{k,m}(\alpha, \mu)} \left( P_1 \right)_{k-1} \quad ; \quad k \geq 2,
\]
where
\[
P_1 = P_1(\beta, \gamma) = \begin{cases} 
\frac{8(1 - \gamma)(\cos^2\beta)}{\pi^2(1 - \beta^2)} & ; \quad 0 \leq \beta < 1 \\
\frac{8}{\pi^2}(1 - \gamma) & ; \quad \beta = 1 \\
\frac{4\beta^2(1 - \gamma)}{\pi^2(1 + \gamma(k^2(t))} & ; \quad \beta > 1
\end{cases}
\]
(8)

For the proof of this theorem, we need the following result by Rogosinski (1943).

Rogosinski’s Theorem. Let \( h(z) = 1 + \sum_{k=1}^{\infty} C_k z^k \) be subordinate to \( H(z) = 1 + \sum_{k=1}^{\infty} C_k z^k \) in \( U \). If \( H(z) \) is univalent in \( U \) and \( H(U) \) is convex, then 
\[
|C_k| \leq |C_1|, \quad k \geq 1.
\]

Proof. Let \( f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in SP_{k,\mu}^{m,\alpha}(\beta, \gamma) \). Then we have
\[
\frac{z(LD_{k,\mu}^{m,\alpha} f(z))'}{LD_{k,\mu}^{m,\alpha} f(z)} < P_{\beta,\gamma}(z).
\]
Define \( q(z) = \frac{z(LD_{k,\mu}^{m,\alpha} f(z))'}{LD_{k,\mu}^{m,\alpha} f(z)} = 1 + \sum_{k=1}^{\infty} C_k z^k \). \( P_{\beta,\gamma}(z) \) is univalent in \( U \) and \( H(U) \) is the conic domain is a convex domain, so, Rogosinski’s theorem applies. Thus, we obtain
\[
|C_k| \leq P_1, \quad k \geq 1
\]
where \( P_1 = P_1(\beta, \gamma) \) is given by (9). By rewriting
\[
q(z) = \frac{z(LD_{k,\mu}^{m,\alpha} f(z))'}{LD_{k,\mu}^{m,\alpha} f(z)} = 1 + \sum_{k=1}^{\infty} C_k z^k.
\]
Define
\[
\frac{k - 1}{k - 1} \sum_{j=0}^{k-1} \left( \lambda \Psi_{m,k}(\alpha, \mu) \right) a_j^2 = \sum_{j=0}^{k-1} \left( \lambda \Psi_{m,k}(\alpha, \mu) \right) a_j^2.
\]
By comparing coefficients of \( z^k \) on both sides, we get
\[
(1 - \lambda)C(m,k) + \lambda \Psi_{m,k}(\alpha, \mu) a_j = \sum_{j=0}^{k-1} \left( \lambda \Psi_{m,k}(\alpha, \mu) \right) a_j^2,
\]
with \( a_1 = 1 \) and therefore,
\[
|a_k| = \frac{\sum_{j=1}^{k-1} \left( \lambda \Psi_{m,k}(\alpha, \mu) \right) a_j^2}{(1 - \lambda)C(m,k) + \lambda \Psi_{m,k}(\alpha, \mu)}
\]
(11)

From (11), when \( k = 2 \), we obtain
\[
|a_2| = \frac{\sum_{j=1}^{k-1} \left( \lambda \Psi_{m,k}(\alpha, \mu) \right) a_j^2}{(1 - \lambda)C(m,2) + \lambda \Psi_{2,m}(\alpha, \mu)}
\]

Therefore, the result is true for \( k = 2 \). Now, let \( k \geq 2 \) and assume that the inequality (8) is true for all \( k \geq j + 1 \) or equivalently \( j \leq k - 1 \). From (11), we get
By using (10) and applying the induction hypothesis to $|a_j|$, we obtain

$$
|p_k| \leq \frac{1}{(k-1)(1-\lambda)C(m,k) + \lambda \psi_{k,m}(\alpha,\mu)} \left[ 1 + \sum_{j=2}^{k-1} \left| \frac{1}{j} \right| B \{ 1 - \lambda C(m,j) + \lambda \psi_{j,m}(\alpha,\mu) \} \right].
$$

Thus, we have

$$
|p_k| \leq \frac{1}{(k-1)(1-\lambda)C(m,k) + \lambda \psi_{k,m}(\alpha,\mu)} \left[ 1 + \sum_{j=2}^{k-1} \left( \frac{1}{j} \right) \right] \text{ for } k \geq 2.
$$

Now we obtain a sufficient condition for $f$ to be in $SP^m_{\lambda, \alpha} (\beta, \gamma)$.

**Theorem 3.** Let the function $f \in A$ of the form (1). If

$$
\sum_{k=2}^{\infty} [k(1 + \beta) - (1 + \gamma)] \left( 1 - \lambda \right) C(m,k) + \lambda \psi_{k,m}(\alpha,\mu) |a_k| \leq 1 - \gamma
$$

then $f \in SP^m_{\lambda, \alpha} (\beta, \gamma)$ where $0 \leq \beta < \infty$, $m \in \mathbb{N}_0 = \{0,1,\ldots\}$, $0 \leq \alpha < 1$, $\lambda \geq 0$, and $\mu \geq 0$.

**Proof.** It suffices to show that

$$
\beta \left| z (LD^m_{\lambda, \alpha} f(z))^\gamma \right| - 1 - \text{Re} \left( \frac{z (LD^m_{\lambda, \alpha} f(z))^\gamma}{LD^m_{\lambda, \alpha} f(z)} - 1 \right) < 1 - \gamma.
$$

We have

$$
\beta \left| z (LD^m_{\lambda, \alpha} f(z))^\gamma \right| - 1 - \text{Re} \left( \frac{z (LD^m_{\lambda, \alpha} f(z))^\gamma}{LD^m_{\lambda, \alpha} f(z)} - 1 \right) \leq (1 + \beta) \left| \frac{z (LD^m_{\lambda, \alpha} f(z))^\gamma}{LD^m_{\lambda, \alpha} f(z)} - 1 \right|
$$

$$
\leq (1 + \beta) \sum_{k=2}^{\infty} \left( 1 - \lambda \right) C(m,k) + \lambda \psi_{k,m}(\alpha,\mu) |a_k| z^k.
$$

The last expression is bounded above by $(1 - \gamma)$ if (12) is satisfied.

**III. GROWTH AND DISTORTION THEOREMS**

In this section we prove some growth and distortion theorems for the class $SP^m_{\lambda, \alpha} (\beta, \gamma)$.

**Theorem 4.** If $f(z) \in SP^m_{\lambda, \alpha} (\beta, \gamma)$ then

$$
r \frac{(1 - \gamma)}{(2 - \gamma + \beta)(1 - \lambda)C(m,2) + \lambda \psi_{2,m}(\alpha,\mu)} \leq \left| f(z) \right| \leq r \frac{(1 - \gamma)}{(2 - \gamma + \beta)(1 - \lambda)C(m,2) + \lambda \psi_{2,m}(\alpha,\mu)} z^2.
$$

where $|z| = r$ with equality for

$$
f(z) = z + \frac{(1 - \gamma)}{(2 - \gamma + \beta)(1 - \lambda)C(m,2) + \lambda \psi_{2,m}(\alpha,\mu)} z^2.
$$

**Proof.** Since $f(z) \in SP^m_{\lambda, \alpha} (\beta, \gamma)$ by applying assertion (12) of Theorem 3, we have

$$
\sum_{k=2}^{\infty} |a_k| \leq \frac{(1 - \gamma)}{2(1 + \beta) - (1 + \gamma)(1 - \lambda)C(m,2) + \lambda \psi_{2,m}(\alpha,\mu)}.
$$

From (1), we have

$$
|f(z)| = \left| z + \sum_{k=2}^{\infty} a_k z^k \right| \leq |z| + \sum_{k=2}^{\infty} |a_k| z^k \leq |z| + \sum_{k=2}^{\infty} |a_k| \leq r + \sum_{k=2}^{\infty} |a_k|,
$$

Then, from (13), we obtain

$$
\sum_{k=2}^{\infty} (k-1)(1-\lambda)C(m,k) + \lambda \psi_{k,m}(\alpha,\mu) |a_k| |z^k| \leq (1 + \beta) \sum_{k=2}^{\infty} \left( 1 - \lambda \right) C(m,k) + \lambda \psi_{k,m}(\alpha,\mu) |a_k| |z^k|.
$$

http://dx.doi.org/10.15242/IIE.E0315028 42
\[ |f(z)| \leq r + r^2 \sum_{k=2}^{\infty} |a_k| \]
\[ \leq r + \frac{1 - \gamma}{[2(1 + \beta) - (\beta + \gamma)](1 - \lambda)C(m,2) + \lambda \Psi_{2,m}^r(\alpha, \mu)} r^2 \]

Also from (1), we have
\[ |f(z)| \geq |z| - \sum_{k=2}^{\infty} |a_k||z|^{k-2} \]
\[ \geq |z| - |z|^2 \sum_{k=2}^{\infty} |a_k| |z|^{k-2} \]
\[ \geq |z| + |z|^2 \sum_{k=2}^{\infty} |a_k| \geq r + r^2 \sum_{k=2}^{\infty} |a_k| \]

Then, from (13), we obtain
\[ |f(z)| \geq r - r^2 \sum_{k=2}^{\infty} |a_k| \]
\[ \geq r - \frac{1 - \gamma}{[2(1 + \beta) - (\beta + \gamma)](1 - \lambda)C(m,2) + \lambda \Psi_{2,m}^r(\alpha, \mu)} r^2 \]
Thus, the proof of Theorem 4 is complete.

Theorem 5. If \( f(z) \in SP_{k,\mu}^n(\beta, \gamma) \) then
\[ 1 - \frac{2(1 - \gamma)}{(2 - \gamma + \beta)[(1 - \lambda)C(m,2) + \lambda \Psi_{2,m}^r(\alpha, \mu)]^r} \leq |f'(z)| \]
\[ \leq 1 + \frac{2(1 - \gamma)}{(2 - \gamma + \beta)[(1 - \lambda)C(m,2) + \lambda \Psi_{2,m}^r(\alpha, \mu)]^r} \]

where \( |z| = r \) with equality for
\[ f(z) = z + (1 - \gamma)
\]
\[ (2 - \gamma + \beta)[(1 - \lambda)C(m,2) + \lambda \Psi_{2,m}^r(\alpha, \mu)]^r z^2 \]

Proof. From the proof of Theorem 4, we have
\[ |f(z)| \leq |z| + |z|^2 \sum_{k=2}^{\infty} |a_k| \leq |z| + \frac{1 - \gamma}{[2(1 + \beta) - (\beta + \gamma)][(1 - \lambda)C(m,2) + \lambda \Psi_{2,m}^r(\alpha, \mu)]^r}|z|^2 \]

Therefore, by letting \( |z| = r \), we have
\[ |f'(z)| \leq 1 + \frac{2(1 - \gamma)}{[2(1 + \beta) - (\beta + \gamma)][(1 - \lambda)C(m,2) + \lambda \Psi_{2,m}^r(\alpha, \mu)]^r} \]
Also from the proof of Theorem 4, we have
\[ |f(z)| \geq |z| - |z|^2 \sum_{k=2}^{\infty} |a_k| \leq |z| - \frac{1 - \gamma}{[2(1 + \beta) - (\beta + \gamma)][(1 - \lambda)C(m,2) + \lambda \Psi_{2,m}^r(\alpha, \mu)]^r}|z|^2 \]

Therefore,