Ant Colony Optimization for Reducing the Consistency Ratio in Comparison Matrix

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***Abstract*—** The acceptable consistent matrix is one of the important issue in comparison matrix to confirm the rational of decision maker opinion. This paper presents a method using ant colony optimization (ACO) to solve the inconsistent comparison matrix to obtain the best consistent ratio (CR), named ACSICR. This method aims to enhance the CR by fractioning the original element to several candidates element to substitute. To evaluate the performance, the ACSICR is tested on some inconsistent comparison matrices. The experimental results have shown that the proposed algorithm is able to decrease the consistent ratio significantly.

***Keywords*—**Ant Colony Optimization, comparison matrix, consistent ratio, analytic hierarchy process

I. Introduction

ONE of the principal concerns in comparison matrix is the consistency. In multi criteria decision making (MCDM), a comparison matrix represents the decision maker (DM) opinion to choose some decision alternatives[1, 2]. However, the comparison matrix which is identified as inconsistent can not be used as judgment. If DM has a opinion that A is better than B, and B is better than C. Then, it can be said as consistent opinion, if the opinion of DM said "A is better than C". Contrarily, it is a inconsistent logic if the opinion of DM is "C is better than A". The value consistency of comparison matrix is set as consistent ratio (CR).

There are two approaches to modify inconsistent matrices into consistent one : (1) The reassessment process of preference DMs to get the new element matrix which is consistent. This approach sometimes need a more time because the reassessment is repeated until the matrices are consistent. (2) Modify the value of preference DMs by a method until fulfilling the threshold consistent ratio. This approach takes attention the researcher to modify the inconsistent comparison matrix [3-10]. Ishizaka and Lusti [3] designed an expert module in some steps starting indicating errors, explain errors and proposes an alternative solutions. Daji et al.[4] repaired the inconsistent comparison matrix by proposed the induced matrix to identify the elements which lead the matrix to be inconsistent. At last, they suggested to alter that elements to get consistent matrix. Therefore, they did not change almost of all the elements matrix except it is suspected to make inconsistent matrix. Jose and Lamata[5] proposed the method to find the random coefficient index instead of Saaty used. They used the simpler function than Saaty to define the accepting or rejecting matrices. They also suggest the levels of consistency to consider restrictive situations. Xu and Wei[6] presented a consistent matrix B by replacing the inconsistent matrix A with $b_{ij}=a^{\alpha}_{ij}\,(W_i/W_j)^{1-\alpha}$, where $W = (W_1...W_i...W_{ij})^T$ is the eigenvector A, and α is a positive value closer 1.0. Cao et al.[7] extended Xu and Wei method by decomposing the original matrix as the Hadamard product of a consistent matrix and a reciprocal deviation matrix. A modified matrix is built via a convex combination of the reciprocal deviation matrix and a zero deviation matrix.

Beside the aforementioned method, there are also modifying the inconsistent matrix using the intelligent algorithm to defeat the inconsistency matrix. Lin et al.[8] and Jose [9] used genetic algorithm (GA) to obtain the consistent matrices. Yang et al.[10] combine particle swarm optimization (PSO) and taguchi method to reduce the inconsistency matrices. The taguchi method is incorporated into the procedure to reduce the number of experiments required for tuning the control parameter of PSO. Besides considering CR must be less than 0.1, Lin et al. and Yang et al. also determine 2 aspects, difference index (*Di*) aspect which represent the distance matrix and consistent ratio aspect. These two aspects are combined in the overall index (*OI*). They proposed method repairing the inconsistent matrix while minimizing the *OI*.

A. Motivation and Contribution

There are many researches for modifying inconsistent comparison matrix with several methods and several aims. However, there is no research emphasizing to get the lowest CR. In fact, sometimes the lowest consistent ratio is needed to ensure the reference is a good logic. The smaller CR indicates the more logical reference to use as decision making.

The aim of this paper is to propose an efficient, simple method to get the smallest of the consistent ratio. To implement this idea, we use ant colony optimization (ACO) which is one of the intelligent algorithm using swarm optimization. ACO was widely used due to its top performing to solve various optimization problems such as traveling salesman problem (TSP) [11, 12], clustering [13], vehicle routing

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problem (VRP)[14], data mining [15], shop scheduling [16] and so forth. The ant algorithm is used to repair the inconsistent matrix to be consistent by modify the discrete value to the continue value.

B. Organization

The remainder of this paper is organized as follows: Section II gives a brief introduction the problem inconsistency matrix in AHP. Section III provides a detailed description of the proposed algorithm. Performance evaluation of the proposed algorithm is presented in Section IV. Conclusion is given in Section V.

II. Related Work

A. Consistency Ratio

Analytic hierarchy process (AHP) is one of tool decision making process to obtain the priority alternative. AHP is represented in a comparison matrix to represent the judgment of decision makers. Element comparison matrix can reflect the subjective opinion that expose strength of the preference and feeling [17]. Saaty [1] proposed 9 scale value (1, 2, 3,...,9) to define the element of comparison matrix. This comparison matrix is stated as a_{ij} which defines the dominance of alternative i over j, where $1<a_{ij}<9$, and $a_{ij}=1/a_{ji}$.

The important in this comparison matrix is satisfying the consistency. The inconsistent matrix can not be used as a good reference to make decision. Saaty [18] defined the consistent as : the intensities of relations among ideas or object based on particular criterion justify each other in some logical way. The consistency ratio (CR) is used to measure the degree of logical respondent opinion. The lower CR is the more logic respondent opinion. The consistency is perfect if CR=0. Saaty defined the threshold of CR is 0.1. The CR which is lower than 0.1 can be categorized as consistent, otherwise is inconsistent. A comparison matrix is stated as perfect consistent if all relationship elements are satisfied as described on Eq. (1).

$$a_{ij} = a_{ik}.a_{kj} \tag{1}$$

Saaty [1] also proposed the method to measure the consistent ratio (CR) which is described on Eq. (2), (3), (4).

$$A \text{ x } W = \lambda_{\max} \text{ x } W, \tag{2}$$

$$CI=(\lambda_{max}- n) / (n-1), \tag{3}$$

$$CR=CI / RI, \tag{4}$$

where λ_{max} is the largest eigenvalue, W is the eigenvector of the matrix. CI is consistent index, n represents number criteria or size matrix, and RI (random consistency index) is the average index of randomly generated weights.

TABLE I
RANDOM CONSISTENCY INDEX (*RI*)

Number criteria	3	4	5	6	7	8	9
RI	0.58	0.9	1.12	1.24	1.32	1.41	1.45

Value RI on each size matrices can be described on Table 1. The perfect consistent is obtained when eigenvalue maximum same with number criteria ($\lambda_{max}= n$). However the perfect consistentcy matrix is hard to achieve especially in the big size matrix.

B. Problem Definition

Since the basic recruitment CR should be less than 0.1, the primary objective is the CR definitely to be minimized. However , to preserve the most of the original, the deviation of modified matrix is limited. Ma[19] proposed the variable, δ and σ to define the deviation two matrices as described on Eq.(5) and Eq.(6).

$$\delta = max_{i,j}\{|a'_{i,j} - a_{ij}|\},\ i, j = 1,2, \dots, n\ ;\ 0 \le \delta < 2\ , \tag{5}$$

$$\sigma = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}(a'_{i,j} - a_{i,j})^2}\ \ / \ n\ , \tag{6}$$

That research concluded that for nine scale, $\delta < 2$ and $\sigma < 1$ are ideally to ensure the deviation still acceptable. Therefore, in this study, besides chasing the lowest consistent ratio, δ and σ also limited less than 2 and 1, respectively

III. Proposed Method

A. The Concept

To apply ACO, elements on comparison matrix must be set as nodes to travel. Due to the value element of PWM is reciprocal ($a_{ij} = 1/a_{ij}$), it can only encode the lower triangular elements of the matrix as nodes[10]. The encoding of matrix A can be assembled by pick row by row sequentially in elements of lower triangular matrix A which is represented as G_A as shown in Fig.1. Ant travels starting from node a_{21} and finishng at node $a_{43,}$ and the number of element G_A is (n^2-$n/2$).

$$A = \begin{bmatrix} 1 & a_{12} & a_{13} & a_{14} \\ a_{21} & 1 & a_{22} & a_{24} \\ a_{31} & a_{32} & 1 & a_{34} \\ a_{41} & a_{42} & a_{43} & 1 \end{bmatrix}$$

$$G_A = a_{21} - a_{31} - a_{32} - a_{41} - a_{42} - a_{43}$$

Fig.1 Encoding of matrix A

If matrix A is identified as inconsistency matrix, the scale value of the each elements original matrix should be changed with new value. The new values are generated by fractioned from original values into several candidate value. The strategy fractioned to build the candidate value is adjusted as follows.

If the origin element has scale value higher than 1, it should be fractioned into candidate value such that either minimal or maximal has the difference from the original lower than 2. Otherwise, it should be fractioned into candidate value between 1/9 and 1. Then, each candidate value is built from minimal to maximal with based on the interval of fraction factor (ψ). In this study, ψ is set 0.2. Table 2 shows the original element fractioned into some candidate element. For the original element lower than 1, the candidate element can be created by determining its reciprocal on Table 2.

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TABLE II
THE ORIGIN ELEMENT IS FRACTIONED INTO SOME CANDIDATE ELEMENT WITH $\Psi = 0.2$

Origin element	Candidate modified
1	1, 1.2, 1.4, 1.6, 1.8, 2.0
1	$\frac{1}{2}, \frac{1}{1.8}, \frac{1}{1.6}, \frac{1}{1.4}, \frac{1}{1.2}, 1$
2	1.2, 1.4,…, 3.8, 4
3	1.2, 1.4,…, 4.8, 5
4	2.2, 2.4,…, 5.8, 6
5	3.2, 3.4,…,6.8, 7
6	4.2, 4.4,…, 7.8, 8
7	5.2, 5.4,…, 8.8, 9
8	6.2, 6.4,…, 8.8, 9
9	7.2, 7.4,…, 8.8, 9

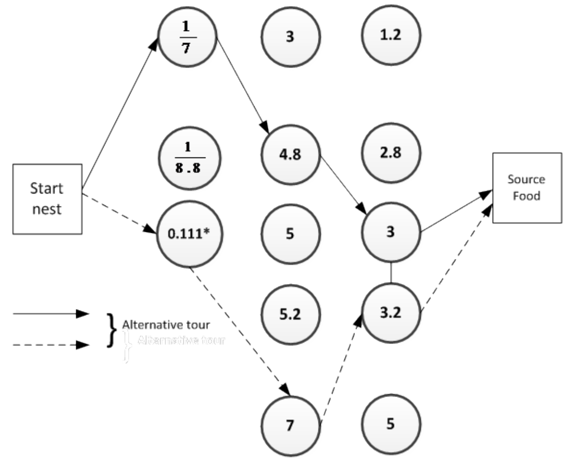

Fig.4 The candidate in each nodes from matrix on Fig. (3).

B. The Algorithm ACSICR

Due to the aim to satisfy the threshold consistent ratio, the first objective is getting the CR less than 0.1. To get this target, the ant moves from the candidate in one node to the candidate in the next node to substitute the origin value. As aforementioned, there are some candidates in each nodes to choose. To give illustration, matrix *A* on Fig.3 (a) which is represented as G_A on Fig.3 (b) is depicted as the tour of ant on Fig.4. The ants start from "nest" stage to search the food by travelling nodes by nodes and finish on "food" stage. On each nodes, ant can choose only one candidate node before continuing to the next node.

$$\begin{matrix} 1.000 & 9.000 & 0.200 \\ 0.111 & 1.000 & 0.333 \\ 5.000 & 3.000 & 1.000 \end{matrix}$$

(b) G_A = 0.111 - 5 – 3.000

Fig.3 (a) Matrix *A* with $n=3$ (b) Encode of matrix *A*

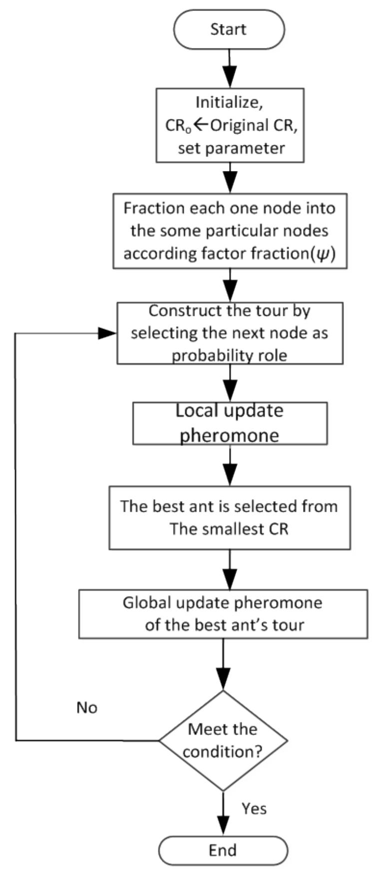

Fig.5 Flowchart of algorithm ACSICR

There are two variables in ACO, pheromone and distance heuristic. Due to each candidate nodes has a same probability to generate the good consistent ratio, the distance heuristic is neglected. Therefore, the value of the distance of one node to next each candidate element will be same.

Fig. 5 shows the algorithm of ACSICR. This figure shows that before constructing the tour, each node will be fractioned into some nodes. To start the tour, each ants is placed in one of the candidate node on the first node (*node1*) randomly. Each ant chooses one of candidate nodes in next nodes with a probability that is function of the amount of pheromone (τ) on the connecting edge before continuing to next node. Suppose the candidate nodes are represented *i* and *j* respectively. While the *k*-th ant is in candidate node *i*, the next candidate node *j* is selected from the the set feasible, $Nk(i)$, according to the two possibilities, P^k_{ij}, as described on Eq. (7) and Eq. (8):

a. if $q \le q_0$ (exploitation) (7)
$j = \arg\max\ [\tau(i,u)^{\alpha}],\ \mu \in j_k(i)$,

b. if $q > q_0$ (exploration) (8)
$P_k(i,j) = \frac{[\tau(i,j)]^{\alpha}}{\sum_{\mu \in j_k(i)} [\tau(i,j)]^{\alpha}}$,

where q is a random number ; q_o is a parameter; τ represents pheromone value. While choosing the node, ant also performs the local update pheromone as described on Eq. (9) :

$$\tau(i,j) = (1-\rho)\tau_{(i,j)} + \rho\Delta\tau_{(i,j)} \tag{9}$$

After all ants travel all nodes, the performance of ants will be evaluated to conclude who the best ant is. The smallest consistency ratio (CR) is chosen as the best ant. The edges of tour of best ant traveled will be updated by the adding the pheromone which called global update pheromone. The pheromone level is updated by applying the updating rule of Eq. (10) and Eq. (11).

$$\tau(i,j) = (1-\rho)\tau_{(i,j)} + \rho\Delta\tau_{(i,j)}, \tag{10}$$

where

$$\Delta\tau(i,j)=\begin{cases}\dfrac{Q*\tau_o}{CR}, & \text{if (i,j)} \in \text{best ant tour}\\ 0, & \text{otherwise}\end{cases} \tag{11}$$

$0<\rho<1$ is the pheromone decay parameter, τ_o is the pheromone initialization and Q is constant.

IV. Experimental Result

A. Parameter and Setting

To evaluate the performance of ACSICR, some parameters are set in this study.

TABLE III
SETTING AND PARAMETER

Parameter	Symbol	Set Value
Number ant	M	10
Iteration	nc	30/200/500
Degree Pheromone	α	2
Initial Pheromone	τ_o	$2/(n^2$-n) ; n=size matrix
Pheromone decay	ρ	0.1
Constant Update	Q	0.001

TABLE IV
THE DATASET INCONSISTENT MATRICES

Matrix	Value	CR
	Size 3x3	
M_1	5; 5-0.2[a]	0.254
M_2	5; 0.333-0.2[a]	0.117
	Size 4x4	
M_3	9; 0.333-0.2; 5-0.5-2[b]	0.172
M_4	0.2; 3-5; 0.25-0.5-0.5[c]	0.191
	Size 5x5	
M_5	3; 0.5-0.143; 6-9-9; 2-4-4-0.2	0.139
M_6	0.333; 0.111-2; 1-7-8; 2-0.5-7-0.5	0.307
	Size 6x6	
M_7	0.5; 9-3; 1-4-0.2; 5-5-5-7; 3-3-0.333-3-0.143	0.136
M_8	5; 3-5; 7-7-3; 5-7-7-3; 3-3-3-3	0.225

The matrices [a], [b],and [c] are picked from [10], [4], and[18]

Table III shows the setting of various parameters for this algorithm. There are 6 various matrices size which some of them are picked from other papers, while some of them are created as shown on Table IV. For small size matrix (4x4), middle size matrix (5x5), big size matrix (6x6) are conducted in 30, 200 and 500 iterations respectively.

B. Performance Analysis of the ACSICR

As aforementioned, the indicator of the best ant is getting the smallest consistent ratio. In the beginning iteration, the ant is not always get the consistent tour. By laying pheromone in each iteration the ant generates the edges preferable in next iteration. In case M_8, Fig.6 shows that no ant (from 10 ants) gets the consistent tour in beginning iteration. However, with more iterations, the number of ants which obtains the threshold CR increases progressively. Ants starts find the consistent tour since iteration 21^{th}. Increasing iterations, the number ants achieves also increases. Furthermore since iteration 256, all ants can obtain the consistent matrix.

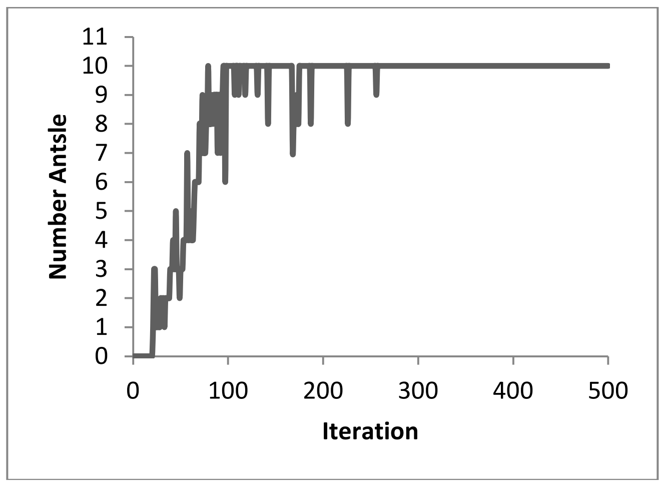

Fig 5.Number ants which get the consistent matrix (case M_8).

The story of convergence can be depicted on Fig.7. In case M_8, the best CR is achieved in 146^{th} iteration with CR=0.0416. The result is stable since the iteration 223^{th}.

Table V shows the performance of ACSICR enhancing the consistent ratio of inconsistent matrix. Some of them in small matrix (M_2, M_3, M_4 and M_6) can obtain almost the perfect consistent matrix. Contrary, in the big size matrix (M_5, M_7 and M_8) can only obtain the CR higher than 0.04.

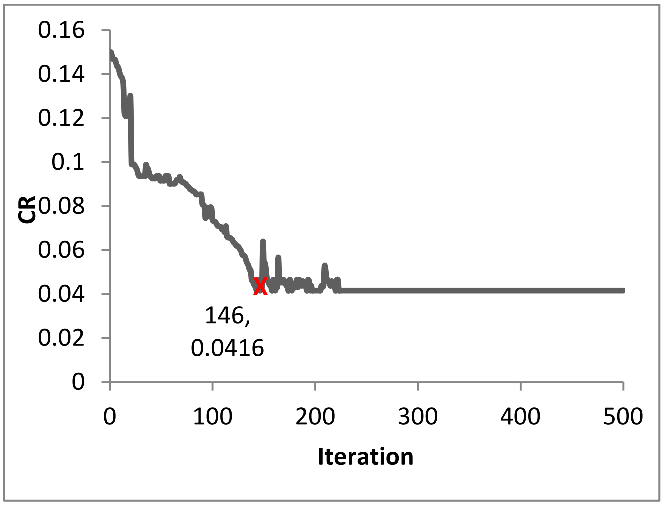

Figure 6. The process of convergence (case M8)

TABLE V
PERFORMANCE ACSICR IN REPAIRING SOME MATRIX

Matrix	Origin and its modified	CR
M_1 n= 3	1.000 0.167 0.333 6.000 1.000 3.000 3.000 0.333 1.000	0.0163
M_2 n=3	1.000 0.2238 1.200 4.200 1.000 5.000 0.833 0.200 1.000	0
M_3 n=4	1.000 0.138 1.000 0.2699 7.200 1.0000 7.000 2.000 1.000 0.143 1.000 0.278 3.800 0.500 3.600 1.000	0
M_4 n=4	1.000 3.000 0.500 2.000 0.333 1.000 0.172 0.625 2.000 5.800 1.000 3.800 0.500 1.600 0.263 1.000	0
M_5 n=5	1.000 1.000 1.800 0.192 0.357 1.000 1.000 5.000 0.135 0.417 0.556 0.200 1.000 0.128 0.208 5.200 7.400 7.800 1.000 3.000 2.800 2.400 4.800 0.333 1.000	0.0472
M_6 n=5	1.000 3.200 8.800 0.625 1.200 0.313 1.000 2.400 0.192 0.417 0.113 0.417 1.000 0.111 0.132 1.600 5.200 9.000 1.000 1.400 0.833 2.400 7.600 0.714 1.000	0.0064
M_7 n=6	1.000 1.000 0.119 0.500 0.185 0.385 1.000 1.000 0.500 0.417 0.161 0.500 8.400 2.000 1.000 3.000 0.333 2.000 2.000 2.400 0.333 1.000 0.143 0.500 5.400 6.200 3.000 7.000 1.000 5.200 2.600 2.000 0.500 2.000 0.192 1.000	0.0520
M_8 n=6	1.000 0.333 0.278 0.135 0.143 0.200 3.000 1.000 0.313 0.185 0.185 0.263 3.600 3.200 1.000 0.500 0.200 0.417 7.400 5.400 2.000 1.000 0.555 0.833 7.000 5.400 5.000 1.800 1.000 0.833 5.000 3.800 2.400 1.200 1.200 1.000	0.0417

V. CONCLUSION

This paper presents a method, ACSICR, to repair inconsistent comparison matrix to obtain the smallest consistent ratio to enhance the logic reference. This method is built based on ACO algorithm, which was successful in solving several optimization problems. In this method, the original element was fractioned to some candidate elements which is chosen by ant to construct the tour. By laying pheromone in edge travelled, ant can find the preferable edge which construct the best consistent ratio. The result of modifying six matrices comparison shows that ACSICR method is effective to enhance the consistent ratio.

At last, this method changes all of the origin value to new value to get the consistent matrix. It is an interesting research in future to implement ACO repairing the inconsistent matrix with target the minimal of number origin element changed.

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