Extremal Solutions for Powered Lunar-Descent Guidance and Targeting

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Abstract**—**A new closed-form solution for the constant thrust arc of the braking phase of a lunar descent trajectory is presented as an enhancement of Apollo ground-based targeting, thereby enabling an Apollo- class explicit real-time on-board targeting procedures. It is shown that integrated real-time guidance and targeting can be formulated entirely by closed-form extremal solutions. The trajectory and guidance performances, simulated using these solutions, have been shown to compare favorably with the reconstructed Apollo 12 descent trajectory.

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Keywords**—** lunar-descent, on-board guidance and targeting, commanded acceleration, optimal control.

I. Introduction

IN this paper, the main elements of the Apollo targeting program associated with the original design of the lunar descent maneuvers are considered. The terminology used is based on the general definitions, conventions and the flow-chart diagrams employed in Apollo missions [1]. The lunar descent and landing was comprised of ballistic, ullage, trim, braking, approach and terminal descent phases. The targeting program that prescribed the maneuvers during the descent was executed on the ground before the mission and provided the initial, terminal, and target states for the braking and approach phases [2]. The results of the targeting programs were placed in the memory of the onboard computer and used to execute the guidance program. The guidance algorithm was executed onboard and at each cycle it provided a re-designated landing site, commanded acceleration, commanded unit thrust and window vectors, updated target referenced time, and updated transformation matrices from the platform system to guidance and body systems. The descent and landing trajectory is nearly planar and significant part of the trajectory is approximated by two quartic polynomials [1]. The approach phase (AP) quartic is defined in a closed form, while the determination of the braking phase (BP) quartic requires iterations in order to satisfy the terminal constraints as the first part of the braking phase has no solution and the state at initial point of the braking phase is unknown. The use of a quartic for the reference trajectory approximation is reasoned by the fact that the formulation is of a minimum order to satisfy the initial, terminal, and operational constraints between the phases. As previously mentioned, the part of the braking phase from the entry orbit to the initial point of the throttle recovery phase had no closed form solution, hence the Apollo ground-based targeting programs used numerical iterations to compute a viable trajectory on the braking phase and to compute the target states for the guidance program [1]. This means that the guidance target states cannot be modified during the actual descent, thereby eliminating the possibility of a practical real-time targeting capability. This capability is extremely important from the point of view of future robotic and human landing missions to the Moon. The benefits of the real-time targeting capability are, in particular, the lander's ability to re-designate the landing site autonomously at any time, to perform a pinpoint landing to conduct potential terrain mapping. Therefore, the derivation of the explicit closed-form solutions for non-zero thrust arcs and their application to powered descent and landing maneuvers would represent a qualitative enhancement of Apollo targeting and guidance programs [3],[4].

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II. Optimal Control Problem and Extremals

A. Problem statement

Assume that the lander's descent propulsion system (DPS) is characterized by the given constant I_{sp} and F_{max}. Then the exhaust velocity of gases, c and the maximum mass-flow rate, β_{max} are

$$c = I_{sp} g_0, \beta_{max} = F_{max}/c \tag{1}$$

Similar to the Apollo DPS constraints, it is assumed that the maximum allowable thrust is not achievable on the descent trajectory. As c=*const*, this means that the mass-flow rate satisfies the condition $\beta < \beta_{max}$. Consider a planar motion of the lander's center of mass in a Newtonian field. If a polar coordinate system $Or\theta$ with the origin at the Moon's center of gravitational attraction, O of a celestial body is introduced, then the spacecraft's motion can be described by the differential equations (Fig.1) [4], [5]:

$$\begin{gathered} \dot{u} = \frac{c\beta}{m} e_r - \frac{\mu}{r^2} + \frac{w^2}{r}, \\ \dot{w} = \frac{c\beta}{m} e_\theta - \frac{uw}{r}, \\ \dot{r} = v, \\ \dot{\theta} = \frac{w}{r}, \\ \dot{m} = -\beta. \end{gathered} \tag{2}$$

Here the variables r, θ, u, w and m will be called the state variables, and are assumed to be real functions of time and

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continuous, but their derivatives may have discontinuities. It is assumed that all variables in Equation (2) are defined on the interval $[t_0, t_1]$. The initial time of flight is fixed, but the final time of motion may be fixed or free. The variables e_r, e_θ and β are considered controls and subject to the constraints:

$$e_r^2 + e_\theta^2 = 1, \;\; 0 \le \beta \le p_1\, \beta_{max}, \tag{3}$$

where $p_1 < 1$ is the constant coefficient of the thrust percentage to be determined. c and β_{max} are known constants and computed using (1). The initial state of the lander is defined by the perilune of an entry orbit with known perilune and apolune altitudes, h_p and h_a, and by the orbital velocity at this point and the initial mass at t_0:

$$r_0, \theta_0, u_0, w_0, m_0 \tag{4}$$

The final state is defined by a point at which the magnitudes of the position and velocity vectors are given:

$$r(t_1) = r_1, \quad v(t_1) = v_1 \tag{5}$$

where $v = \sqrt{u^2 + w^2}$. The performance index or the functional J of the problem is defined as

$$J = m_0 - m_1 \tag{6}$$

where $m_1 = m(t_1)$. The functional J and its high-order derivatives are considered to be continuous on $[t_0, t_1]$. Note, that in general, \$J\$ can be implicit or explicit function of t_0 and t_1. Then the problem is to find an optimal control and an optimal trajectory, which satisfy (2)-(5) and transfer the lander from the initial state to the final state while minimizing J. Note that if $p_1 = 1$, that is if F_{max} is achievable, then the problem (2)-(6) is a typical Mayer's variational (or optimal control) problem.

B. Extremals of the problem

It can be shown that the analysis of the first-order necessary conditions of optimality and the Weierstrass condition in the problem (2)-(6) lead to the following conclusions [3],[5]:

(1) An optimal trajectory may contain a combination of a null-thrust arc, where $\beta = 0$ and $\chi \le 0$, an intermediate-thrust arc, where $0 < \beta < p_1\, \beta_{max}$ and $\chi \equiv 0$, and constant-thrust arc, where $\beta = p_1\, \beta_{max}$ and $\chi \ge 0$. Here $\chi = \frac{c}{m}\lambda - \lambda_m$ is the switching function, and λ and λ_m are the primer vector magnitude and the Lagrange multiplier associated with the mass;

(2) On the optimal trajectory, the thrust direction cosines must satisfy the conditions:

$$e_r = \frac{\lambda_u}{\lambda}, \quad e_\theta = \frac{\lambda_w}{\lambda}, \tag{7}$$

where $\lambda = \sqrt{\lambda_u^2 + \lambda_w^2}$ is known as the magnitude of the primer vector, $\boldsymbol{\lambda}$. These conclusions allow us to express (2) and the first-order necessary conditions of optimality by the canonical equations formed separately for each thrust arc mentioned above and the transversality conditions

$$\lambda_{s1} = -\nu \frac{\partial J}{\partial x_s} = 0, \quad s = 1, \dots, \le 5, \tag{8}$$

where x_s is any state variable not specified at t_1. For a normal solution, without loss of generality, one can use $\nu = 1$ in (8). Taking into account (7) the constant thrust (CT) arc can be described by the canonical equations [4]:

$$\begin{aligned}
\dot{u} &= \frac{c\beta}{m}\frac{\lambda_u}{||\lambda_v||} - \frac{\mu}{r^2} + \frac{w^2}{r},\\
\dot{w} &= \frac{c\beta}{m}\frac{\lambda_w}{||\lambda_v||} - \frac{uw}{r},\\
\dot{r} &= u,\\
\dot{\theta} &= \frac{w}{r},\\
\dot{m} &= -\beta,\\
\dot{\lambda_u} &= \lambda_w \frac{w}{r} - \lambda_r,\\
\dot{\lambda_w} &= -2\lambda_u \frac{w}{r} + \lambda_w \frac{u}{r} - \frac{\lambda_\theta}{r},\\
\dot{\lambda_r} &= \lambda_u\left(\frac{w^2}{r^2} - 2\frac{\mu}{r^3}\right) - \lambda_w \frac{uw}{r^2} + \lambda_\theta \frac{w}{r^2},\\
\dot{\lambda_\theta} &= 0,\\
\dot{\lambda_m} &= \frac{c\beta}{m^2}\sqrt{\lambda_u^2 + \lambda_w^2},
\end{aligned} \tag{9}$$

with the Hamiltonian

$$H = \lambda_u \dot{u} + \lambda_w \dot{w} + \lambda_r \dot{r} + \lambda_\theta \dot{\theta} + \lambda_m \dot{m}\,, \tag{10}$$

where u, w and λ_u, λ_w are the components of $\boldsymbol{v}$ and $\boldsymbol{\lambda}_v$ respectively, and $\lambda_r, \lambda_\theta$ and λ_m are the Lagrange multipliers related to r, θ and m, and β is determined using the sign of χ. As is known from calculus of variations, a trajectory satisfying (9) is said to be an extremal of the problem and an optimal trajectory with minimum J is one of the members of the family of extremals that also satisfy (4), (5) and (8). Due to the fact that the first part of the BP was flown with a nearly constant thrust, and the thrust on an intermediate-thrust arc is variable, this paper deals only with the extremal solutions for the CT arcs with $\beta = p_1\, \beta_{max}$, $p_1 < 1$ and which satisfy (4), (5), (8) and (9). It should be noted here that for various maneuvers, the extremal trajectories, which satisfy the boundary and transversality conditions, have been shown to be very close to optimal trajectories [4]. For these arcs, the following integrals of (9) are known [4]:

$$H = C, \;\; \lambda_\theta = \lambda_{\theta 0}, \;\; m = m_0 - \beta t\,, \tag{11}$$

where C, $\lambda_{\theta 0}$ and m_0 are the integration constants. The first expression of (11) is known as the Hamiltonian integral which takes place due to an implicit independence of H on time, t. Similarly, the integral for λ_θ is known as the cyclic integral as H does not explicitly depend on θ.

III. Modification And Extension Of Braking-Phase Targeting

As mentioned in the "Introduction", the BP has two parts. The first part, known as the pre-quartic part, starts at the perilune of the entry orbit and ends at the TCR point. The second part, the quartic part, ends at the initial point of the AP. The pre-quartic part of the phase will be investigated in the next subsections. The quartic formulation allows us to satisfy five constraints on the reference trajectory [1]. As is known, the main part of the descent trajectory consists of three phases: braking, approach and terminal descent. Each of the iterations in the BP-targeting routine requires the execution of the AP-targeting routine in order to match the terminal conditions of the BP and the initial conditions of the AP, and to find the right conditions at the TCR point. Note that (a) the number of iterations is unknown, and their convergence is not guaranteed in advance for arbitrary final conditions for both AP and BP, and (b) the actual thrust is not the same as the commanded thrust until the TCR point is reached. These two factors show the importance of describing the entire BP by a closed-form solution to avoid the iterations and repeated runs of the AP-targeting routine. As the last portion of the BP is quartic, a closed-form solution is needed to describe the part of the BP. This, in particular, demonstrates the utility of the development of the proposed CT arc solutions to the guidance and targeting.

C. Closed-form solutions for braking phase

From the engineering point of view, it is possible to align the thrust vector opposite to velocity vector for descent and landing purposes. In particular, when non-zero final velocity is desired, such an alignment of the thrust vector can be used on the portion of CT arc of the descent trajectory. Below it will be shown that this approach to determine the thrust direction enables the integration of (9) in elementary functions except for the polar angle, which is expressed in quadratures. The corresponding solutions for CT arc are then used to replace the first part of the BP of the landing trajectory. Based on the assumptions about the thrust vector's alignment, consider the expression

$$\boldsymbol{\lambda}_{\mathbf{v}} = -q(t)\mathbf{v}, \tag{12}$$

where the primer vector, $\boldsymbol{\lambda}_{\mathbf{v}}$ determines the thrust direction, *q(t)* is the unknown function, continuous and differentiable with respect to time. It will be shown below that $q>0$ and $\dot{q} > 0$ at any time on the CT arc that satisfy the boundary conditions (4) and (5). The scalar form of (12) is written as

$$\lambda_u = -q(t)u,\ \ \lambda_w = -q(t)w. \tag{13}$$

Taking into account that $\lambda_u = \lambda\ \sin\varphi$ and $\lambda_w = \lambda\ \cos\varphi$ where $\lambda = \|\boldsymbol{\lambda}\|$ and φ is the thrust angle, defined as the angle between $\boldsymbol{\lambda}$ and perpendicular to ***r***, from (13) it follows that

$$u = w\ \tan\varphi. \tag{14}$$

This equation shows that the thrust direction is collinear to the velocity direction (see (12)).

If ψ is the angle between the thrust direction and the reference axis is $\theta = 0$, then

$$\psi = \theta - \varphi + \tfrac{\pi}{2}. \tag{15}$$

To summarize the solution process, described in this subsection, one can write the solutions of (9) in the form of functions in terms of q, v and φ, which are the functions of time:

$$\begin{aligned} u &= -v\sin\varphi,\\ w &= -v\cos\varphi,\\ r &= -\tfrac{a_1}{qv\cos\varphi},\\ \theta &= \tfrac{1}{a_1}\Theta + \theta_0,\\ m &= m_0(1-at),\\ \lambda_u &= qv\sin\varphi,\\ \lambda_w &= qv\cos\varphi,\\ \lambda_r &= -2\Lambda + \lambda_{r0},\\ \lambda_\theta &= \lambda_{\theta 0},\\ \lambda_m &= \frac{c\beta}{m_0^2}\int_{t_0}^{t}\frac{qv}{(1-at)^2}dt + \lambda_{m0}, \end{aligned} \tag{16}$$

where

$$\begin{aligned} \Theta &= \int_{t_0}^{t} q\, v^2(1-\frac{4}{3}(\cos(\frac{\eta}{3}))^2)dt,\\ q &= a_2\,\frac{(1-ht)^{k_1}}{(1+at)^{k_2}},\\ v &= v_0\,\frac{(1-ht)}{(1+at)},\\ \cos\eta &= -\sqrt{27}\tfrac{{a_1}^2}{2\mu q^2 v}\left(\tfrac{\beta}{m_0(1-at)} + 2\tfrac{\dot{q}}{q} - \tfrac{\ddot{q}}{\dot{q}}\right),\\ \Lambda &= \int_{t_0}^{t} q\, v^2\left(\frac{\beta}{m_0(1-at)} + 2\frac{\dot{q}}{q} - \frac{\ddot{q}}{\dot{q}}\right)dt,\\ \sin\varphi &= \tfrac{2}{\sqrt{3}}\cos\left(\tfrac{\eta}{3}\right),\\ k_1 &= \frac{a\alpha}{h}\frac{h+k}{a-h},\qquad k_2 = \frac{a+k}{a-h}, \end{aligned} \tag{17}$$

and the complete set of the integration constants of (16) are $a_1, a_2, k, h, v_0, \theta_0, m_0, \lambda_{r0}, \lambda_{\theta 0}\,(=0)$ and λ_{m0}.

This completes the derivation of the solutions for the CT arc. Validity of (16) with (17) has been verified by substituting them into (9) and obtaining the identities for each equation of this system. No singularities may occur in (16) with (17) as

$$q>0,\ \dot{q} > 0,\ \varphi = \tfrac{\pi}{2}, \tfrac{3\pi}{2}, \ldots,\ a \neq h,\ a < 1/t_1,\\ k_1 \neq 0,\qquad k_2 \neq 0.$$

D. Initial and final conditions for braking phase

In this phase, the first part of the BP will be called the BP-CT arc as it will satisfy the final conditions and constraints of the BP. This arc starts at the perilune of the entry orbit and ends at the TCR point. It can be shown that

$$\varphi_0 = \pi,\ \tfrac{a_1}{a_2} = -\sqrt{\mu p},\ v_0 = \sqrt{\tfrac{\mu}{p}}\,(1+e),\ h = \alpha h - k, \tag{18}$$

where $\varphi_0 = \varphi(t_0)$, and the entry orbit parameters can be computed as

$$e = \frac{h_a - h_p}{h_a + h_p + 2r_p},\ \ p = \frac{(r_p + h_p)(h_a - h_p)}{h_a + h_p + 2r_p}$$

This selection of the constants $\varphi_0, \frac{a_1}{a_2}$, v_0 and h satisfies the continuity conditions for the state vector at a point that connects the perilune of the entry orbit with the BP-CT arc. If the mass m_{bf} at the BP-final point, BPF is given, then one can show that the time t_1 spent on the BP-CT arc can be found using the formulae:

$$t_1 = \frac{1}{\beta}(m_0 - m_{bf} - \bar{\beta}\ \Delta t_{cf}),\quad \bar{\beta} = p_2\beta_{max},$$
$$\beta = p_1\beta_{max}, \tag{19}$$

where Δt_{cf} is the flight time on the BP-quartic, $\bar{\beta} = |\dot{m}_{bf}|$ and $p_2 = [0.11, \sim 0.65]$. Taking into account that $r_1 = r(t_1)$ and $v_1 = v(t_1)$ are given at the TCR point, it can be shown that

$$h = \frac{1+\alpha a\, t_1 - \delta}{t_1(1-\delta)},\quad \cos\varphi_1 = -\frac{\sqrt{\mu p}}{r_1\, v_1 q_1}, \tag{20}$$

and $\delta = \frac{v_0}{v_1}$. The constant a_2 can be found using the transversality condition and the condition that $\chi = 0$ at the end point of the BP-CT arc. Given p_2 and m_{bf} at the initial point of the BP-quartic, the thrust percentage, p_2 on the BP-CT arc can be computed using the transcendental equation

$$1 - \frac{\mu p}{r_1^2 v_1^2\, q_1^2} = \frac{r_1^4 v_1^2}{\mu^2}\left(\frac{p_2\beta_{max}}{m_0(1-at)} + 2\frac{\dot{q}}{q} - \frac{\ddot{q}}{\dot{q}}\right). \tag{21}$$

Evaluated at t_1, (21) establishes an important relationship between the final position and velocity vectors, and the thrust percentage, which corresponds to the thrust level at the initial point of the BP-quartic and the current mass. Once p_1 is determined, the mass-flow rate on the BP-CT arc is computed using (19). In the case, when the thrust percentage p_2 is given (for instance, the thrust percentage on Apollo pre-quartic BP was 93.8 %, that is p_1=0.938), the time t_1 can be found from (21) and then p_2 and m_{bf} can be determined from (19). This completes the computation of the integration constants and the unknown parameters of the BP-CT arc.

E. Continuity of position and velocity vectors

The continuity conditions at the TCR point are formed by equating the corresponding components of the final position and velocity vectors of the BP-CT arc defined in the $X_p\, Z_p$-plane, which contains the Moon center of mass, the lander's center of mass and the nominal landing site [1]. Assume that the position, velocity and thrust acceleration vectors at the BP-final point are given as

$$r_{bf}\ (r_{xf}, r_{yf}, r_{zf}),\ v_{bf}\ (v_{xf}, v_{yf}, v_{zf}),$$
$$a_{bf}\left(a_{xf}, a_{yf}, a_{zf}\right) = \frac{c\beta}{m_{bf}}\boldsymbol{u_F} + \boldsymbol{g_F}, \tag{22}$$

where $\boldsymbol{u_F} = (\sin\psi_F, 0, \cos\psi_F)$ and $\boldsymbol{g_F} = (g\ \cos\theta_F, 0, \sin\theta_F)$ are the unit thrust and gravity vectors at the point being considered, ψ_F is the angle between the thrust vector and local horizontal and given according to the Apollo targeting constraints, and

$$g = \frac{\mu}{|r_{bf}|^2},\quad \theta_F = \tan^{-1}\frac{r_{xf}}{r_{zf}}. \tag{23}$$

Here and below, the subscript f will mean the values of the corresponding parameters at the final point of the corresponding phase. Then the continuity conditions at this point can be written in the following form:

$$x = r\sin\theta_2 = r_{xf} + v_{xf}T + a_{xf}\frac{T^2}{2} + j_{xf}\frac{T^3}{6} + s_{xf}\frac{T^4}{24},$$
$$z = r\ \cos\theta_2 = r_{zf} + v_{zf}T + a_{zf}\frac{T^2}{2} + j_{zf}\frac{T^3}{6} + s_{zf}\frac{T^4}{24},$$
$$u\ \sin\theta_2 + w\ \cos\theta_2 = v_{xf} + a_{xf}T + j_{xf}\frac{T^2}{2} + s_{xf}\frac{T^3}{6}, \tag{24}$$
$$u\ \cos\theta_2 - w\sin\theta_2 = v_{zf} + a_{zf}T + j_{zf}\frac{T^2}{2} + s_{zf}\frac{T^3}{6},$$

where *x*, *z*, *u* and *w* are the components of the position and velocity vectors computed at the final point of the BP-CT arc (the TCR point), θ_2 is the polar angle of this point, and *T* is the target referenced time spent on the BP-quartic trajectory. The target referenced time, *T* is assumed to be given (typically, 120 seconds in Apollo 11 and 12 missions). The variable *T* should not be confused with the subscript *T*. These equations allow us to find the unknowns $j_{xf}, j_{zf}, s_{xf}, s_{zf}$ in terms of θ_2. Consequently, the quartic state at the TCR point can be determined completely by using the position and velocity vectors defined at the same point of the BP-CT arc.

These procedures show that no repeated AP- and BP-targeting and state vector iterations are needed to construct the BP extremal trajectory. This is the significant advantage of the proposed solutions in the design of a powered descent trajectory as an alternative to the Apollo-era trajectory and guidance design.

IV. Simulations

Studies have been conducted to analyze the CT arc that would transfer the lander from the entry orbit to the throttle recovery point followed by the throttle recovery phase which ends at the terminal point of the braking phase. All values for the position, velocity, mass, mass-flow rate, exhaust velocity and time are typical or have been taken from the Apollo guidance performances [1], [2]. The following parameters used in the simulations:. The entry orbit is characterized by *e*=0.0290778 and *p*=1803973 m. The orientation of the orbit is determined by the solution process yielding $\omega = 1.973$ rad. The engine parameters are *c* = 3150 m/s and β_{max} = 16.6 kg/s. The initial mass and mass estimate at the terminal point of the braking phase are $m_0 = 14800$ kg and m_{brf} = 8670 kg. The altitude of the initial point of the braking phase and altitude of apogee of the entry orbit are h_0 = 15000 m and h_a=120000 m. The values shown above are typical and do not represent any specific mission. The lander trajectory and the thrust percentage on the CT arc are presented in figures 2 and 3.

The studies included three parts of the trajectory: 1) from entry orbit to throttle recovery point [1]; 2) from the throttle recovery point to the initial point of the approach phase (BP-quartic); and 3) from initial point of the approach phase to the initial point of terminal descent phase (AP- quartic). In the simulations the trajectories started from different altitudes using different entry orbits. It should be noted that the entry orbit, initial altitude, initial mass and mass flow-rate, altitude and velocity at the throttle recovery point, exhaust velocity and percentage of thrust should satisfy the transcendental equation in (21). The following set of parameters has been selected to analyze the relationship between the parameters by employing (21):

Set 1: e=0.0266312, p= 1799876 m, ω = 1.973 rad (Orbit 1), h_0= 15186.4 m, h_a =111120 m, r_1 =1743213 m, m_0 =15121 kg, m_{brf} = 8662.6 kg, c = 3150 m/s, and β_{max} = 16.6 kg/s.

This set of data mostly corresponds to Apollo 12 characteristics and partially used by Yang [2], [6]. The results of the simulations show that for a given altitude r_1 at the throttle down point there exists a certain set of values of $v_1$with corresponding duration of CT and percentage of thrust (see Fig.4). For the altitudes selected, the thrust percentage can vary from 63% to 154%. Analysis show the necessity of additional studies in order to determine the regions of permissible values for all other main parameters mentioned above. Unlike the CT arc, the BP- and AP- quartics have not changed their shape, including monotonic character of the altitude on these quartics. The qualitative behavior of the velocity and acceleration vectors, and the thrust percentage with respect to duration of CT and final velocity at throttle recovery point remained same compared to the analysis with the previous sets of the initial conditions.

V. Conclusions

The Apollo targeting program and the proposed enhancement through modification to the braking phase of the lunar descent maneuver are described. The part of the braking phase starting from the entry orbit and ending with the initial point of the throttle recovery was described by an analytic, closed-form extremal CT arc solution. This modification avoids the iterations and convergence issues, and explicitly solves the BP and AP targeting problems, thereby enhancing the original Apollo targeting program. The closed-form solution proposed enables the onboard computer to perform real-time targeting at any time during the descent as the development of the ground-based targeting usually performed before the mission. The corresponding targeting algorithm can be shown to be robust to arbitrary initial conditions, feasible, structurally simple and allows straightforward execution at any given time.

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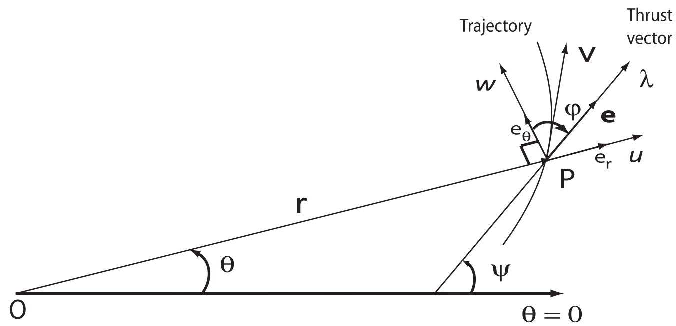

Fig. 1 The polar coordinate system and vectors used.

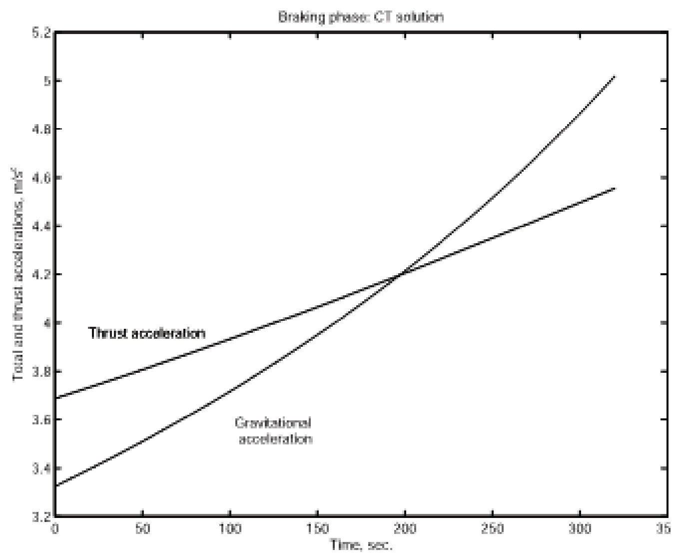

Fig. 2 Thrust and gravitational accelerations

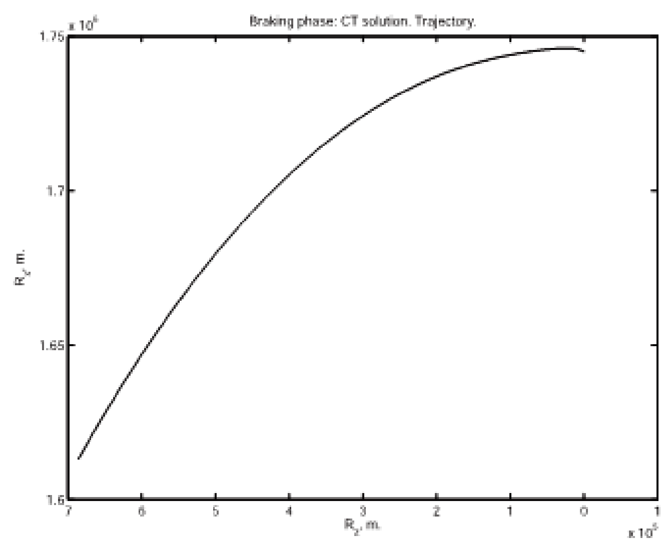

Fig. 3 Components of the position vector

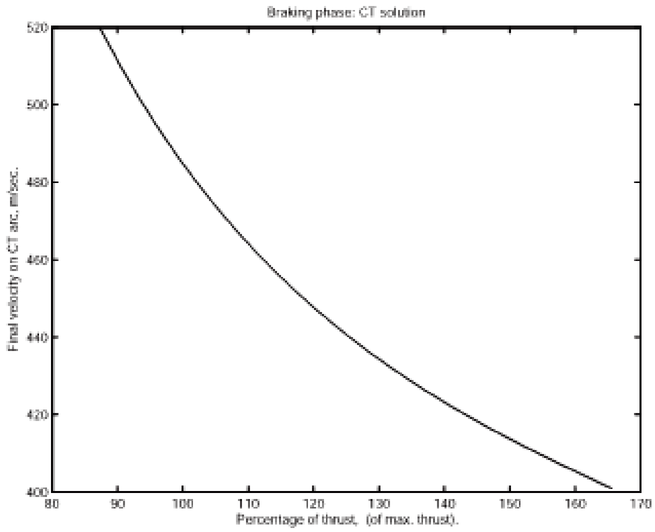

Fig. 4 Thrust percentage on CT arc. Orbit 1

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