Equilibrium Analysis of a Four Strut Tensegrity Mechanism

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***Abstract*—**This paper presents a case where the equilibrium configurations of a compliant tensegrity mechanism are analyzed. The mechanism consists of four struts, four elastic ties, and eight non-elastic ties. The tensegrity is defined as being in a state of equilibrium when the following four conditions are met: the sum of the forces at each of the top four coordinate points of the tensegrity are zero, the forces in each member are equal in magnitude, opposite in sense, and collinear. The input values that are defined include the length of each strut, the length of each elastic tie, the bottom four coordinate points, and an initial guess of the top four coordinate points. Mathematical optimization via the interior point method is used to minimize a function that defines the orientation of the tensegrity closest to equilibrium. In the case that is presented all the spring constants associated with the elastic ties remain constant and the tensegrity configuration closest to equilibrium is found. A contribution of this paper is to show how certain design restrictions affect the equilibrium solution configuration.

***Keywords*—** Tensegrity, Equilibrium Configuration.

I. Introduction

The term tensegrity is one that was created by Buckminster Fuller as a contraction of the words 'tensional integrity' (Fuller, 1975). Integrity as it relates to structures refers to the stability of a system. In the case of a tensegrity system, forces are allowed to be transferred from one part of the structure to another through compressive members and tension members (Pugh, 1976). These compressive members have internal forces that are in compression and are labeled as struts. The tension members have internal forces in tension and are labeled as ties. These components make up the tensegrity system which is a system that is in a stable self-equilibrated state comprised of a set of compressed components inside a continuum of tensioned components (Motro, 2003). This self-equilibrated state is what makes the structure of a tensegrity unique. As a result of a tensegrity being in a state of stable equilibrium, the structure will return to the original given configuration after the application of small perturbations anywhere within the configuration (Skelton and de Oliveira, 2009).

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There have been several significant applications of tensegrity research spanning over many different fields. An American cell biologist named Donald Ingber worked with biologist Dimitrije Stamenovic to develop a mathematical model of tensegrity that predicts how cells from many different tissues behave mechanically (Stamenovic, 1996). More recently bioengineers Hod Lipson and Radhika Nagpal used tensegrities to construct macroscale robots that are able to move individually as well as self-assemble into larger collective mechanisms (Paul et al 2006).

II. Problem Statement

The configuration of the tensegrity that is analyzed here is shown in Figure 1. The top and bottom of the tensegrity consists of a total of eight points. The bottom of the device is labeled as points O1, P1, Q1 and R1. The top of the device is labeled as points O2, P2, Q2 and R2. There are a total of twelve ties that include four elastic ties and eight non-elastic ties. The elastic ties fall along points P1O2, Q1P2, R1Q2, and O1R2. The other 8 ties are non-elastic and make up the bottom and top of the tensegrity. For this analysis it is assumed that the four base points, O1, P1, Q1, and R1 are fixed. The problem at hand is to determine the equilibrium configuration of the mechanism when given the base point locations, the strut and tie lengths, and the spring constants and free lengths of the elastic ties.

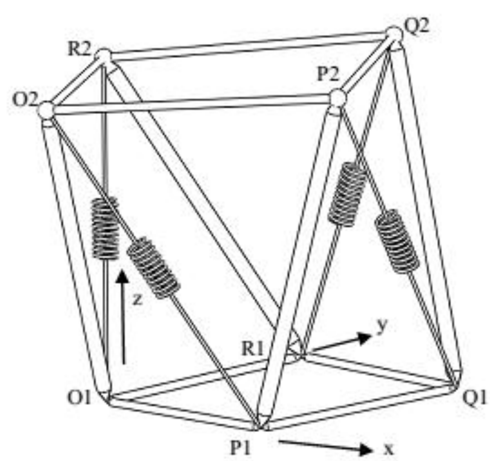

Fig. 1: 4-strut Tensegrity Mechanism

Each of the eight points illustrated on the tensegrity in Figure 1 represents components with x, y, and z coordinates. Since only the top four points are allowed to vary, the tensegrity configuration that is closest to equilibrium is defined by the twelve coordinates of these four points. A large scale iteration approach is undertaken in order to obtain the equilibrium configuration. To increase efficiency in solving this problem, the twelve output variables can be reduced to eight by representing each of the top four coordinate points by rotation angles, since the length of each strut is known. These rotation angles are defined in the following manner: each of the four struts are aligned along the x axis and then rotated about the z axis and the y axis respectively as the bottom four points remain constant. The rotation angle about the z axis is defined as alpha and the rotation about the modified y axis is defined as beta. Figure 2 illustrates how this rotation is done with strut O. The bottom point O1 remains fixed and the top point is allowed to move as the rotation angles vary.

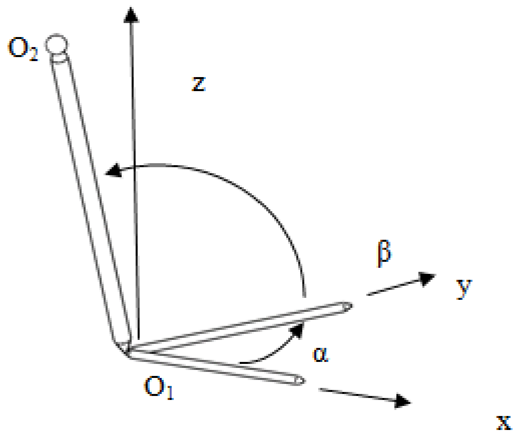

Fig. 2: Rotation angles shown on Tensegrity

The transformation of any point in one coordinate system to a reference coordinate system can be found when the relative position and orientation of the pair of coordinate systems are known (Crane and Duffy, 1998). Compound transformations define instances where multiple translations and rotations can take place to define the initial and final coordinates of a given point. Equation 1 defines this transformation for the top four points as follows:

$$^{A}P_1 = T_{Tr}*T_\alpha*T_\beta*{}^{B}P_1 \tag{1}$$

AP1 and BP1 represent the end points of the strut members. BP1 is the variable that defines the top four points before the rotations and AP1 defines the top four points seen in Figure 1 after the rotations. In Equation (1) TTr, Tα, Tβ are 4x4 transformation matrices. TTr represents the translation that occurs to go from the origin to each of the bottom four points on the tensegrity. In Figure 1 the origin is defined as the point O1. This transformation matrix is defined as.

$$T_{Tr} = \begin{vmatrix} 1 & 0 & 0 & B_x \\ 0 & 1 & 0 & B_y \\ 0 & 0 & 1 & B_z \\ 0 & 0 & 0 & 1 \end{vmatrix} \tag{2}$$

Here the variables Bx, By and Bz are the x, y and z coordinates of the bottom four points. Since the origin lies at O1 this matrix will be equal to the identity matrix for the first point. The second two matrices Tα and Tβ are defined as:

$$T_\alpha = \begin{vmatrix} \cos(\alpha) & -\sin(\alpha) & 0 & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \tag{3}$$

$$T_\beta = \begin{vmatrix} \cos(\beta) & 0 & \sin(\beta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \tag{4}$$

Now given any two points along each end of the struts, the rotation angles about the y and z axis can be solved for.

The configuration of the tensegrity that is the closest to equilibrium is solved for the following case: where the forces in the elastic ties are constants defined by constant ‘k’ values. The problem statement is presented as follows:

Given:
1. O1, P1, Q1 and R1 (bottom four coordinate points)
2. L1, L2, L3 and L4 (length of the four strut members)
3. k1, LO1 (spring constant and free length for elastic tie 1)
4. k2, LO2 (spring constant and free length for elastic tie 2)
5. k3, LO3 (spring constant and free length for elastic tie 3)
6. k4, LO4 (spring constant and free length for elastic tie 4)
7. O2, P2, Q2, and R2 (initial guess for the top four coordinate points)

Find: The tensegrity configuration that is the closest to equilibrium.

III. Solution

The solution for the stated problem was solved for using a combination of systems of equations involving the forces of the tensegrity members, the given point coordinates, the rotation angles, as well as mathematical optimization using the interior point algorithm. The members of the tensegrity that cannot change in length include the struts and the non-elastic ties. The lengths of the struts are defined by first stating the numerical coordinates of the bottom four points and the top four points. The bottom four points will remain constant throughout the problem as the top four points will be allowed to change as long as each of the strut lengths remain constant. The forces in the struts and non-elastic ties are labeled as constant variables and the numerical values for each of the forces in the elastic ties are entered using the following equation for force in a spring:

$$F_{spring} = k*(L-L_O) \tag{5}$$

Then the sum of forces at each of the top coordinate points of the tensegrity is taken. Figure 3 illustrates the sum of the forces at the top point O2 of the tensegrity. The sum of forces at the other three top points are defined in the same manner.

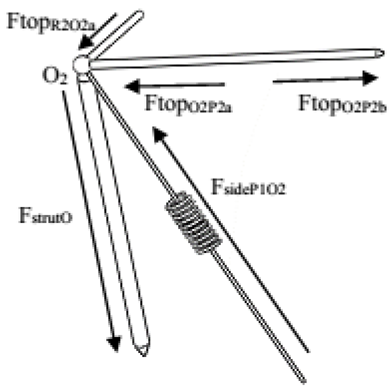

Fig. 3: Sum of Forces at Point O2

The force in the top tie is in tension and therefore the force vectors are moving away from each other along the same axis the tie lies on. This satisfies the first two conditions for equilibrium which state that internal forces have to be opposite in sense and collinear (Seely and Ensign, 1921). The next property that needs to be satisfied for the mechanism to be in equilibrium is the forces in each of the members have to be equal in magnitude (Hibbler, 2013). The struts and elastic ties' internal forces are defined as uniform and therefore satisfy this last condition of equilibrium. Therefore if the internal forces of the top ties are equal in magnitude this condition for equilibrium will be satisfied.

To ensure the internal forces of the top ties are equal in magnitude a function was devised to minimize the difference between the internal forces in each of the top ties. This mathematical minimization problem was created in Matlab using the interior point method algorithm. First a function f(x) is created as a function of the vector 'x' and corresponding equality and inequality constraints are defined. The vector x in this problem is defined as a design vector for one case that will be solved for. This case will output a design vector that includes the eight rotation angles that describe the orientation of the tensegrity. The function defines the position of the tensegrity in terms of the sines and cosines of the rotation angles making it a nonlinear optimization problem. The interior point algorithm can used to minimize a non-linear function subject to equality and non-equality constraints (Bonnans et al, 2006). The Matlab optimization toolbox is used to carry out this minimization that will drive the difference between the internal forces in the top ties of the tensegrity to zero and therefore satisfying the third condition for equilibrium.

The internal forces in the top ties are labeled as $Ftop_{O2P2a}$, $Ftop_{O2P2b}$, $Ftop_{R2O2a}$, $Ftop_{R2O2b}$, $Ftop_{P2Q2a}$, $Ftop_{P2Q2b}$, $Ftop_{Q2R2a}$, and $Ftop_{Q2R2b}$. The subscript notation denotes the two points at each end of the top tie. Figure 3 shows how the internal forces in the top tie between points O2 and P2 are oriented. The forces in the three remaining side elastic ties are defined in the same way and are labeled as: $F_{sideQ1P2}$, $F_{sideR1Q2}$ and $F_{sideO1R2}$. The variables used to define each of those forces include the following: k_{QP} = 2 lbf/in, LO_{QP} = 1 in, k_{RQ} = 1.8 lbf/in, LO_{RQ} = 1 in, k_{OR} = 1.3 lbf/in, LO_{OR} = .5 in. The forces in each of the four struts are constants denoted using similar notation and are labeled as: F_{strutO}, F_{strutP}, F_{strutQ} and F_{strutR}.

Now all the tensegrity's forces and corresponding variables are defined and the summation of forces is used to define the forces in the struts and elastic ties in terms of the given point coordinates and the forces in the side elastic ties that have been defined as constants. By using the force components in the x, y and z coordinates there will be three equations and three unknowns at each of the top four points of the tensegrity. The three unknowns will include the two forces in the top ties and the force in the strut at each point. In the system of equations at point O2 the three unknowns are $F_{topO2P2a}$, $F_{topR2O2a}$ and F_{strutO}. These three equations and three unknowns are then solved for using MATLAB and expressed in terms of the top coordinates of the tensegrity and the variable $F_{sideP1O2}$ which has already been defined in terms of given numerical values. These steps are repeated for the system of equations associated with the remaining three top points P_2, Q_2 and R_2. Now all of the forces in the struts and the top ties are expressed in terms of the top coordinates of the tensegrity and the forces in the side ties. Defining the forces in this way ensures that the summation at each of the top points of the tensegrity is zero and satisfies the fourth and last condition for equilibrium.

To define an ideal starting position for the coordinates of the top points the motion analysis in the 3D CAD software SolidWorks is used. Figure 4 represents a tensegrity model in SolidWorks that is used in order to get the orientation of a four strut tensegrity in equilibrium. In this model the members are fixed to each of the eight points on the bottom and top of the tensegrity. Numerical values for the 'k' constants and free lengths are entered into the motion analysis toolbox. These values are defined as the following:

O1= [0,0,0]
P1 = [6,0,0]
Q1 = [5,4,0]
R1 = [2,5,1]

The values for the lengths of the struts O, P, Q and R are 8in, 7in, 10in and 7in respectively. The lengths of the top ties going from R_2O_2, O_2P_2, P_2Q_2 and Q_2R_2 are 5in, 7in, 8in and 6 in respectively.

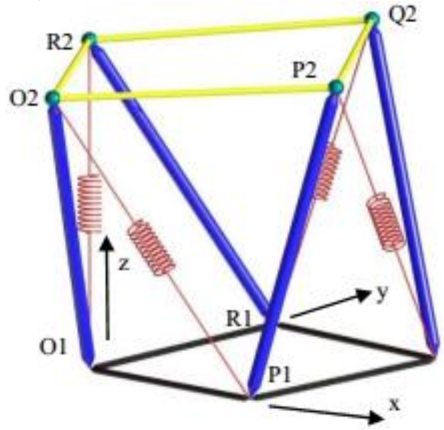

Fig. 4: Rendered 4 Strut Tensegrity Solidworks Image

After numerical values were entered in for the top four points, a damper was selected for each of the four elastic ties so ensure motion of the springs would eventually stop during the motion analysis. The motion study in Solidworks was then run and the values for the top four points of the tensegrity in equilibrium were as follows:

O2 = [6.635038, 0.175616, 4.466031]
P2 = [3.043715, 5.843685, 2.472191]
Q2 = [-2.995701, 1.546081, 5.481518]
R2 = [1.942664,-1.416448,3.797843]

Now that all variables associated with the tensegrity have been defined in terms of constants and the x, y and z coordinates of the top and bottom points, the coordinates are then written in terms of the rotation angles alpha and beta. The angles alpha1 and beta1 correspond to point O_2, alpha2 and beta2 correspond to point P_2, alpha3 and beta3 correspond to point Q_2 and alpha 4 and beta4 correspond to point R_2. For the purpose of the design vector in the Matlab function, α1, β1, α2, β2, α3, β3, α4, β4 correspond to x(1) through x(8) respectively.

Through minimization the sum of two variables, named $Total_{summation}$ and $Total_{difference}$ are going to be driven to zero.

$$F_{value} = Total_{summation} + Total_{difference} \tag{6}$$

These variables represent the sum of the forces at each of the top four points and the difference between the internal forces in each of the top four ties respectively. The output will be the configuration with the function value closest to zero defined by eight rotation angles. The case presented will show the resulting configuration as the forces in the springs remain constant

IV. Results

This case presents results for two sets of constraints. For the first set of constraints, the strut and top ties are restricted to the values of length already defined for them and the beta angles are restricted to the range of 0 to 180 degrees. Here the resulting function value analyzed is calculated from Equation 6 and comes out to a value of 12.492 lbf. The individual values defined in the function value equation are:

$$Total_{summation} = 1.8383e\text{-}07 \text{ lbf} \tag{7}$$
$$Total_{difference} = 12.4920 \text{ lbf} \tag{8}$$

Figure 5 below shows the resulting configuration of the tensegrity plotted in a three dimensional x, y and z plane in Matlab. The blue members represent the struts, the red members represent the elastic side ties, the yellow members represent the top ties, and the black members represent the bottom of the tensegrity fixed to points O_1, P_1, Q_1 and R_1.

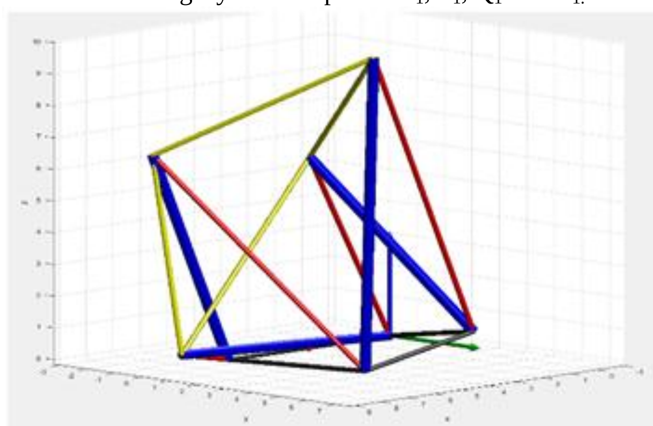

Fig. 5: Tensegrity with set 1 constraints

This resulting configuration shows that one of the strut members is lying almost flat along the x-y plane. To prevent resulting cases like this the next set of constraints will address this design issue by altering the range of degrees for the beta angles. For the second set of constraints the beta angles are restricted to a smaller range of 35 to 180 degrees while the constraints of length used in the previous case remain the same. Here the resulting function value is 20.522 lbf. The individual values defined in the function value equation are:

$$Total_{summation} = 6.7223 \text{ lbf} \tag{9}$$
$$Total_{difference} = 13.8003 \text{ lbf} \tag{10}$$

Here we can see that running the optimization problem with a stricter range for the beta angle values increases the overall function value by almost a factor of two. Figure 6 shows the resulting configuration.

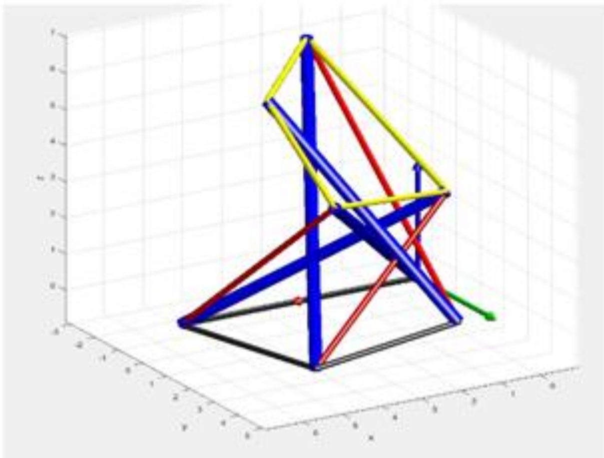

Fig. 6: Tensegrity with set 2 constraints

This figure shows that the design of the tensegrity is improved but at the cost of a much higher function value.

V. Conclusion

This paper presented a case where the orientation of a tensegrity closest to equilibrium was defined and solved for subject to several design constraints. A tensegrity's orientation was defined as being in a state of equilibrium when the sum of the forces at the top of the mechanism is zero as well as the internal forces in the tensegrity's members were all collinear, opposite in sense and equal in magnitude. Running an optimization problem in which the difference between the internal forces in the members as well as the sum of the forces at each of the top points was minimized, the orientation closest to equilibrium was found. Design constraints including the length of the members and the range for the rotation angles were placed on the function being minimized. By analyzing the resulting configurations it was evident that the more design constraints placed on the problem, the farther away from equilibrium the tensegrity mechanism was.

Acknowledgments

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