Magnetostatic Field Calculations Associated with Thick Solenoids in the Presence of Iron Using an Integral Formula Derived in Terms of the Quaternion Variable and the Euler-Maclaurin Summation Formula.

Vasos Pavlika

***Abstract*—**The effect of iron on the uniformity of the field produced by an axisymmetric thick solenoid is considered. Using an integral equation derived for brevity using the quaternion variable of Hamilton the components of the magnetic induction are computed. The solution to the vector potential and field components is also derived using the Euler-Maclaurin Summation formula to convert the doubly infinite summation to an integral.

***Keywords*—** Time independent field, the quaternion variable, the Euler-Maclaurin summation formula.

I. Introduction

THE complex form of Green's theorem is:

$$\oint_C f(z,\bar{z})dz = 2i\iint_R \frac{\partial f}{\partial \bar{z}}dxdy$$

where $f(z,\bar{z})$ is a complex valued function that depends on z and its conjugate $\bar{z}$, with z = x + iy. The region R is bounded by the curve C where first order derivatives of are assumed continuous. Introducing a simple pole in R at z_0 and imposing that $f(z,\bar{z})$, has unit residue, then by construction:

$$f(z,\bar{z}) = w(z,\bar{z})g(z,z_0)$$

Where g has unit residue at $z = z_0$ and $w(z,\bar{z})$ is analytic in R. By enclosing the singularity in a circle $\sum$ centre z_0 and with the usual connecting contour, then for this punctured region as shown in figure 1,

$$\oint_{C'} g(z,z_0)w(z,\bar{z})dz = \oint_C g(z,z_0)w(z,\bar{z})dz$$

$$-\oint_\Sigma g(z,z_0)w(z,\bar{z})dz$$

$$\to \oint_{C'} g(z,z_0)w(z,\bar{z})dz - 2\pi i w_0 \text{ as } \Sigma \to 0$$

so that

$$2\pi i w_0 = \oint_C g(z,z_0)w(z,\bar{z})dz - 2i\iint_R g(z,z_0)\frac{\partial w}{\partial \bar{z}}dxdy$$

(1)

If the pole lies on the curve C then it can be shown using the Plemelj formulae, or by indenting the contour, that

$$\pi i w_0 = \oint_C g(z,z_0)w(z,\bar{z})dz - 2i\iint_R g(z,z_0)\frac{\partial w}{\partial \bar{z}}dxdy$$

Using this equation one would be able to solve a variety of potential rewritten as Fredholm integral equations of the first and second kind respectively.

II. The Three Dimensional Counterpart

Here the four dimensional quaternion operator of Hamilton will be used to derive the three dimensional counterpart of equation (1). The quaternion variable:

$[t, \underline{r}] = (t, x, y, z) = (t, ix, jy, kz)$

where $\underline{r} = \hat{i}x + \hat{j}y + \hat{k}z$, the separators i, j, k satisfy the following multiplication table:

Table I

*	i	j	k
i	-1	k	-j
j	-k	-1	i
k	j	-i	-1

Where the * denotes multiplication, the separators must not be confused with the unit vectors $\hat{i},\hat{j},\hat{k}$. Writing the quaternion v as $[V, \underline{v}]$ where V is a scalar and $\underline{v}$ is the vector $\underline{v} = \hat{i}v_1 + \hat{j}v_2 + \hat{k}v_3$, then it follows that

Vasos Pavlika is with SOAS, University of London, UK, Email: vp4@soas.ac.uk.

$vw = [V, \underline{v}][W, \underline{w}]$

$= [VW - \underline{v}.\underline{w}, V\underline{w} + W\underline{v} + \underline{v} \wedge \underline{w}]$

Where the . and $\wedge$ are the usual scalar and cross product of vectors. Thus for the vector operator $\underline{\nabla}$, it follows that

$$\int_R [0, \underline{\nabla}][V, \underline{v}]dv = \int_S [0, \underline{s}][V, \underline{v}]ds$$

i.e.

$$\int_R [-\underline{\nabla}.\underline{v}, V\underline{\nabla} + \underline{\nabla} \wedge \underline{v}]dv = \int_S [-\underline{n}.\underline{v}, V\underline{n} + \underline{n} \wedge \underline{v}]ds \tag{2}$$

Equation (2) is remarkable in the sense that this simple looking expression contains Gauss' divergence theorem and Stoke's theorem of vector calculus, where $\underline{n}$ is the unit outward normal to the surface S, enclosing the region R and dv and ds are the usual volume and surface differentials respectively. Equation (2) forms the basis of the three dimensional counterpart of equation (1). The functions involved will be dependent on two position vectors $\underline{P}$ and $\underline{Q}$, with W chosen to represent the reciprocal of the distance from $\underline{P}$ to $\underline{Q}$, so that

$$W = \underline{r}^{-1}$$

and

$$\underline{\nabla}_P W = -\underline{r}\underline{r}^{-3},$$

So that $\underline{\nabla}_P^2 W = \underline{\nabla}_Q^2 W = 0$, where differentiation is carried out with respect to the coordinates of $\underline{P}$ with $\underline{Q}$ fixed denoted by the subscript p. When differentiating with respect to the coordinates of $\underline{Q}$ a suffix Q will be used. Now it can be shown that:

$$\underline{[0, \nabla]}[0, W\underline{w}] = [0, \underline{\nabla}][W, 0][0, \underline{w}]$$
$$= [W, 0][0, \underline{\nabla}][0, \underline{w}] + [0, \underline{\nabla}][W, 0][0, \underline{w}]$$

integrating this last identity over a region R, bounded externally by a closed surface S and internally by a small sphere s_0 of radius r_0 and centre Q then

$$\int_{S+s_0} [0, \underline{n}][0, W\underline{w}]ds = \int_R [W, 0][0, \underline{\nabla}][0, \underline{w}]dv - \int_R [0, \underline{\nabla}_Q][W, 0][0, \underline{w}]dv \tag{3}$$

Applying the operator $[0, \underline{\nabla}_Q]$ to both sides of equation (3) gives the left hand side as

$$\int_{S+s_0} [0, \underline{\nabla}_Q][0, \underline{n}][W, \underline{0}][0, \underline{w}]ds$$
$$= \int_{S+s_0} [0, \underline{\nabla}_Q][W, \underline{0}][0, \underline{n}][0, \underline{w}]ds$$
$$= -\int_{S+s_0} [0, \underline{\nabla}W][0, \underline{n}][0, \underline{w}]ds$$
$$= \int_S [0, \underline{rr}^{-3}][0, \underline{n}][0, \underline{w}]ds - \int_{s_0} [0, \underline{nr}_Q^{-2}][0, \underline{n}][0, \underline{w}]ds$$
$$= \int_S [0, \underline{rr}^{-3}][0, \underline{n}][0, \underline{w}]ds + [0, 4\pi \underline{w}_0]$$

as $r_Q \to 0$

similarly for the right hand side of equation (3) i.e.,

$$\int_R [0, \underline{\nabla}_Q][W, 0][0, \underline{\nabla}][0, \underline{w}]dv - \int_R [\underline{\nabla}_Q^2, 0][W, 0][0, \underline{w}]dv$$
$$= \int_R [0, \underline{\nabla}_Q W][0, \underline{\nabla}][0, \underline{w}]dv - \int_R [\underline{\nabla}_Q^2 W, 0][0, \underline{w}]dv$$
$$= \int_R [0, \underline{rr}^{-3}][0, \underline{\nabla}][0, \underline{w}]dv$$

hence by equating both sides the following is valid

$$[0, 4\pi \underline{w}_0] + \int_S [0, \underline{rr}^{-3}][0, \underline{n}][0, \underline{w}]ds = \int_R [0, \underline{rr}^{-3}][0, \underline{\nabla}][0, \underline{w}]dv \tag{4}$$

the vector part of equation (4) is the three dimensional counterpart of (1).

III. Application to Magnetostatic Field Problem

Equation (4) will now be applied to calculate the field components associated with an axisymmetric conductor of rectangular cross section situated equidistant from two semi-infinite regions of iron of finite permeability are computed. The magnetostatic field associated with iron-free axisymmetric systems has been considered in [1], [2] and by many others, for example reference [3] - [5] take into account the effects of the presence of iron on such systems. The main advantages of introducing iron are:

i. Higher fields are provided for the same current, producing substantial power savings over conventional conductors.

ii. The field uniformity is improved even for superconducting solenoids by placing the iron in a suitable position.

The geometry considered is shown in figure 2, a toroidal conductor V' of rectangular cross section having inner radius A, outer radius B and length L-2ε, is located equidistant between two semi-infinite regions of iron of finite permeability a distance L apart, the axis of the torus being perpendicular to the iron boundaries. The region V between the conductor and the iron is assumed insulating. Cylindrical polar coordinates *(ρ,ϕ,z)* are used where ρ and z are normalized in terms of L. Prior to the work described in [3] the presence of iron in axisymmetric systems had been largely ignored see [2] and [6] et al. Using cylindrical coordinates (ρ,φ,z), for the conductor of figure 2 in the presence of iron of finite permeability, the vector part of equation (4) is:

$$4\pi\underline{B}_0(\underline{P})=\int_S\left\{\frac{\underline{r}}{|\underline{r}|^3}\wedge(\underline{n}\wedge\underline{B})-\frac{(\underline{n}.\underline{B})}{|\underline{r}|^3}\underline{r}\right\}ds+\int_V\left\{\frac{\underline{r}}{|\underline{r}|^3}\wedge(\underline{\nabla}\wedge\underline{B})-\frac{(\underline{\nabla}.\underline{B})}{|\underline{r}|^3}\underline{r}\right\}dv \quad (5)$$

The governing equations are those of Maxwell thus:

$$\underline{\nabla}\wedge\underline{B}=\begin{cases}0 \;\; in \;\; V\\ -Ce_\phi \;\; in \;\; V'\end{cases}$$

Where e_ϕ is a unit vector in the direction of increasing ϕ and C is a constant with

$\underline{\nabla}.\underline{B}=0$ in V and V'

With boundary conditions

$$\underline{n}\wedge\underline{B}=\underline{0} \;\; on \;\; z=0,1$$
$$as \;\; \rho\to\infty$$

as $\rho\to\infty,(M\in\Re)$

The position vector of a point $\underline{r}$ in cylindrical coordinates is:

$$\underline{r}=(z-z')\hat{i}-x\sin\vartheta\hat{j}+(\rho-x\cos\vartheta)\hat{k}$$
$$and$$
$$|r|^3=\left((z-z')^2+x^2+\rho^2-2x\rho\cos\vartheta\right)^{3/2}$$

Considering the volume integral over V' in equation (5) and calculating the triple cross product gives

$$\underline{r}\wedge\underline{\nabla}\wedge\underline{B}=$$
$$\mu_0 j\left((x-\rho\cos\vartheta)\hat{i}+(z-z')\sin\vartheta\hat{j}+(z-z')\cos\vartheta\hat{k}\right)$$

and hence the volume integral becomes

$$\mu_0 j\int_{\rho_a}^{\rho_b}\int_{z_a}^{z_b}\int_0^{2\pi}\frac{x(x-\rho\cos\vartheta)dxdz'd\vartheta}{((z-z')^2+x^2+\rho^2-2x\rho\cos\vartheta)^{3/2}}\hat{i}+$$
$$\mu_0 j\int_{\rho_a}^{\rho_b}\int_{z_a}^{z_b}\int_0^{2\pi}\frac{x(z-z')\sin\vartheta dxdz'd\vartheta}{((z-z')^2+x^2+\rho^2-2x\rho\cos\vartheta)^{3/2}}\hat{j}$$
$$+\mu_0 j\int_{\rho_a}^{\rho_b}\int_{z_a}^{z_b}\int_0^{2\pi}\frac{x(z-z')\cos\vartheta dxdz'd\vartheta}{((z-z')^2+x^2+\rho^2-2x\rho\cos\vartheta)^{3/2}}\hat{k}$$

the $\hat{j}$ component vanishes due to the integrand being an odd function, so that

$$\int_V\left\{\frac{\underline{r}}{|\underline{r}|^3}\wedge(\underline{\nabla}\wedge\underline{B})\right\}dv=$$
$$\mu_0 j\int_{\rho_a}^{\rho_b}\int_{z_a}^{z_b}\int_0^{2\pi}\frac{x(x-\rho\cos\vartheta)dxdz'd\vartheta}{((z-z')^2+x^2+\rho^2-2x\rho\cos\vartheta)^{3/2}}\hat{i}+$$
$$\mu_0 j\int_{\rho_a}^{\rho_b}\int_{z_a}^{z_b}\int_0^{2\pi}\frac{x(z-z')\cos\vartheta dxdz'd\vartheta}{((z-z')^2+x^2+\rho^2-2x\rho\cos\vartheta)^{3/2}}\hat{k} \quad (6)$$

Expanding these integrals in a Maclaurin series in ρ it can be shown (see [9]) that the i and j components of expression (6) are given by

$$\mu_0 j\pi\left[\left[\begin{array}{l}2x(w^2+x^2)^{1/2}-\rho\dfrac{(x^2+2w^2)}{xw(w^2+x^2)^{1/2}}\\ +\dfrac{\rho^2}{2!}\dfrac{(x^4+6x^2w^2+4w^4)}{xw(w^2+x^2)^{3/2}}\end{array}\right]_{\rho_a}^{\rho_b}\right]_{z_a}^{z_b}+O(\rho^3)\text{ a}$$

and

$$\frac{\rho\mu_0 j\pi}{2}\left[\left[-\log\left|\frac{(x^2+w^2)^{1/2}-x}{(w^2+x^2)^{1/2}+x}\right|-\frac{2x^2w}{xw(w^2+x^2)^{3/2}}\right]_{\rho_a}^{\rho_b}\right]_{z_a}^{z_b}$$
$$+O(\rho^3)$$

Where w=z-z'. Now to consider the surface integral S where

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$$S=\int_{S_1}\left\{\frac{\underline{r}}{|\underline{r}|^3}\wedge(\underline{n}\wedge\underline{B})-\frac{(\underline{n}.\underline{B})}{|\underline{r}|^3}\underline{r}\right\}ds+\int_{S_2}\left\{\frac{\underline{r}}{|\underline{r}|^3}\wedge(\underline{n}\wedge\underline{B})-\frac{(\underline{n}.\underline{B})}{|\underline{r}|^3}\underline{r}\right\}ds+\int_{S_3}\left\{\frac{\underline{r}}{|\underline{r}|^3}\wedge(\underline{n}\wedge\underline{B})-\frac{(\underline{n}.\underline{B})}{|\underline{r}|^3}\underline{r}\right\}ds\equiv S_1+S_2+S_3.$$

Where the discs S_i (i=1,2,3) are shown in figure 3. On the discs S_1 and S_2, $\underline{n}\wedge\underline{B}=\underline{0}$, so that the integral S_1 becomes

$$S_1=\int_{S_1}\left\{\frac{\underline{r}}{|\underline{r}|^3}\wedge(\underline{n}\wedge\underline{B})-\frac{(\underline{n}.\underline{B})}{|\underline{r}|^3}\underline{r}\right\}ds=-\int_{S_1}\frac{(\underline{n}.\underline{B})}{|\underline{r}|^3}\underline{r}ds,$$

With z=0 on S_1. Using the expression for $\underline{r}$ with outward drawn normal to S_1 equal to –i, then

$$S_1=-\lim_{\rho\to\infty}\left\{\int_{z_a}^{z_b}\int_0^{2\pi}B_z\left(\frac{z'\hat{i}+x\sin\vartheta\hat{j}-(\rho-x\cos\vartheta)\hat{k}}{(z'^2+x^2+\rho^2-2x\rho\cos\vartheta)^{3/2}}\right)\rho d\vartheta dz'\right\}$$

similarly S_2 (with z=1) becomes

$$S_2=-\lim_{\rho\to\infty}\left\{\int_{z_a}^{z_b}\int_0^{2\pi}B_z\left(\frac{(1-z')\hat{i}+x\sin\vartheta\hat{j}-(\rho-x\cos\vartheta)\hat{k}}{((1-z')^2+x^2+\rho^2-2x\rho\cos\vartheta)^{3/2}}\right)\rho d\vartheta dz'\right\}$$

in the integrals S_1 and S_2 the j component vanishes. Similarly on S_3 considering S_3 such that

$$S_3=\int_{S_3}\left\{\frac{\underline{r}}{|\underline{r}|^3}\wedge(\underline{n}\wedge\underline{B})-\frac{(\underline{n}.\underline{B})}{|\underline{r}|^3}\underline{r}\right\}ds$$

and using the vector identity:

$$\underline{A}\wedge(\underline{B}\wedge\underline{C})=(\underline{A}.\underline{C})\underline{B}-(\underline{A}.\underline{B})\underline{C}$$
$$\Rightarrow\underline{r}\wedge(\underline{n}\wedge\underline{B})-(\underline{n}.\underline{B})\underline{r}=(\underline{r}.\underline{B})\underline{n}-(\underline{r}.\underline{n})\underline{B}-(\underline{n}.\underline{B})\underline{r}\tag{7}$$

With outward drawn normal to S_3 radial, so that

$\underline{n}=\sin\vartheta\hat{j}+\cos\vartheta\hat{k}$, equation (7) gives

$$\begin{pmatrix}B_z(-(\rho-x\cos\vartheta)\cos\vartheta+x\sin^2\vartheta)\\-B_\rho\cos\vartheta(z-z')\end{pmatrix}\hat{i}+\begin{pmatrix}B_z\sin\vartheta(z-z')\\+B_\rho\sin\vartheta((\rho-x\cos\vartheta)+x\sin\vartheta\cos\vartheta)\end{pmatrix}\hat{j}+\begin{pmatrix}B_z\cos\vartheta(z-z')\\+B_\rho(\cos\vartheta(\rho-x\cos\vartheta)+x\sin^2\vartheta)\end{pmatrix}\hat{k}$$

so that

$$S_3=\lim_{x\to\infty}\left\{\begin{pmatrix}B_z(-(\rho-x\cos\vartheta)\cos\vartheta+x\sin^2\vartheta)\\-B_\rho\cos\vartheta(z-z')\end{pmatrix}.\left(\frac{\rho d\vartheta dz'}{R^{3/2}}\right)\hat{i}\right\}$$
$$+\lim_{x\to\infty}\left\{\begin{pmatrix}B_z\sin\vartheta(z-z')\\+B_\rho\sin\vartheta((\rho-x\cos\vartheta)+x\sin\vartheta\cos\vartheta)\end{pmatrix}.\left(\frac{\rho d\vartheta dz'}{R^{3/2}}\right)\hat{j}\right\}$$
$$+\lim_{x\to\infty}\left\{\begin{pmatrix}B_z\cos\vartheta(z-z')\\+B_\rho(\cos\vartheta(\rho-x\cos\vartheta)+x\sin^2\vartheta)\end{pmatrix}.\left(\frac{\rho d\vartheta dz'}{R^{3/2}}\right)\hat{k}\right\}$$

therefore

$$4\pi\underline{B}_0(\underline{P})=$$
$$\hat{i}\mu_0 j\pi\left[\left[\begin{matrix}2x(w^2+x^2)^{1/2}-\rho\dfrac{(x^2+2w^2)}{xw(w^2+x^2)^{1/2}}\\+\dfrac{\rho^2}{2!}\dfrac{(x^4+6x^2w^2+4w^4)}{xw(w^2+x^2)^{3/2}}\end{matrix}\right]_{\rho_a}^{\rho_b}\right]_{z_a}^{z_b}+$$
$$\frac{\hat{j}\rho\mu_0 j\pi}{2}\left[\left[-\log|\frac{(x^2+w^2)^{1/2}-x}{(w^2+x^2)^{1/2}+x}|-\frac{2x^2w}{xw(w^2+x^2)^{3/2}}\right]_{\rho_a}^{\rho_b}\right]_{z_a}^{z_b}$$
$$-\lim_{\rho\to\infty}\left\{\int_{z_a}^{z_b}\int_0^{2\pi}B_z\left(\frac{z'\hat{i}+x\sin\vartheta\hat{j}-(\rho-x\cos\vartheta)\hat{k}}{(z'^2+x^2+\rho^2-2x\rho\cos\vartheta)^{3/2}}\right)\rho d\vartheta dz'\right\}$$
$$-\lim_{\rho\to\infty}\left\{\int_{z_a}^{z_b}\int_0^{2\pi}B_z\left(\frac{(1-z')\hat{i}+x\sin\vartheta\hat{j}-(\rho-x\cos\vartheta)\hat{k}}{((1-z')^2+x^2+\rho^2-2x\rho\cos\vartheta)^{3/2}}\right)\rho d\vartheta dz'\right\}$$
$$+\lim_{x\to\infty}\left\{\begin{pmatrix}B_z(-(\rho-x\cos\vartheta)\cos\vartheta+x\sin^2\vartheta)\\-B_\rho\cos\vartheta(z-z')\end{pmatrix}.\left(\frac{\rho d\vartheta dz'}{R^{3/2}}\right)\hat{i}\right\}$$
$$+\lim_{x\to\infty}\left\{\begin{pmatrix}B_z\sin\vartheta(z-z')\\+B_\rho\sin\vartheta((\rho-x\cos\vartheta)+x\sin\vartheta\cos\vartheta)\end{pmatrix}.\left(\frac{\rho d\vartheta dz'}{R^{3/2}}\right)\hat{j}\right\}$$
$$+\lim_{x\to\infty}\left\{\begin{pmatrix}B_z\cos\vartheta(z-z')\\+B_\rho(\cos\vartheta(\rho-x\cos\vartheta)+x\sin^2\vartheta)\end{pmatrix}.\left(\frac{\rho d\vartheta dz'}{R^{3/2}}\right)\hat{k}\right\}$$
$$+O(\rho^3)\tag{8}$$

In this last equation the point $\underline{P}$ is allowed to occupy the boundary points $\underline{Q}$ giving rise to a diagonally dominant system of algebraic equations for the unknown field values B_z and B_ρ on the iron boundaries. Once these have determined, to calculate the field components off the iron boundaries at (ρ, z) the coordinates are input into expression (8) used as a formula. The respective field components $B_z(\rho, z)$ and $B_\rho(\rho, z)$ can then be determined near the axis.

IV. Calculation of the Field Components using the Euler-Maclaurin Summation formula

Here use of the Euler-Maclaurin summation will be made to convert the doubly infinite summation corresponding to the image coils to an integral. Much literature exists on the derivation of the formula thus only the final formula will be quoted. We have seen [10] that:

$$A_\phi(r,z) = \frac{\mu_0 j}{4\pi} \sum_{n=-\infty}^{\infty} K^{|n|} \int_a^b \int_0^{2\pi} \int_\varepsilon^{1-\varepsilon} \frac{x \cos\vartheta dx d\vartheta dz'}{\{(z-z'-n)^2 + r^2 + x^2 - 2xr\cos\vartheta)^{1/2}}$$

and considering the summation first i.e. defining

$$S = \sum_{n=-\infty}^{\infty} \frac{\gamma K^{|n|}}{((\alpha-n)^2+\beta^2)^{1/2}}$$

Where

$$\gamma = x\cos\vartheta, \beta^2 = r^2 + x^2 - 2xr\cos\vartheta \ and \ \alpha = z - z'$$

so that

$$S = \sum_{n=0}^{\infty} \frac{\gamma K^{|n|}}{((\alpha-n)^2+\beta^2)^{1/2}} + \sum_{n=0}^{\infty} \frac{\gamma K^{|n|}}{((\alpha+n)^2+\beta^2)^{1/2}} - \frac{\gamma}{(\alpha^2+\beta^2)^{1/2}}$$

which may be written as

$$S = \sum_{n=0}^{\infty} f_1(n) + \sum_{n=0}^{\infty} f_2(n) - \frac{\gamma}{(\alpha^2+\beta^2)^{1/2}}, say$$

$$= \sum_{n=0}^{\infty} f(n) - \frac{\gamma}{(\alpha^2+\beta^2)^{1/2}}$$

Where $f(n) = f_1(n) + f_2(n)$. So that the effect of the image coils has been separated from the main coil. To these images we apply the Euler-Maclaurin Summation formula. Considering the term

$$\sum_{n=0}^{\infty} f_1(n) = \int_0^\infty f_1(n)dn + \frac{1}{2}\left[f_1(0) - f_1(\infty)\right]$$
$$+ \frac{1}{12}\left[f_1^{'}(\infty) - f_1^{'}(0)\right] - \frac{1}{720}\left[f_1^{'''}(0) - f_1^{'''}(\infty)\right] + \ldots$$

Letting

$$I_1(\alpha) = \int_0^\infty f_1(n)dn = \int_0^\infty \frac{\gamma k^n}{((\alpha-n)^2+\beta^2)^{1/2}} dn$$
$$= \int_0^\infty \frac{\gamma e^{-\delta n}}{((\alpha-n)^2+\beta^2)^{1/2}} dn$$

Where $\delta = \log_e |\frac{1}{K}|$ and $K = \frac{\mu - 1}{\mu + 1}, \mu \neq 1$

So clearly the method will cater for the case when $\mu \neq 1$ but this is as expected as this is the iron free situation. In order to make any progress with this integral the integrand will be expanded in a Maclaurin series in α which will be a small parameter. Thus

$$I_1(\alpha) = I_1(0) + \alpha I_1^{'}(0) + \frac{\alpha^2}{2!} I_1^{''}(0) + O(\alpha^3) \quad (9)$$

So that

$$I_1(0) = \int_0^\infty \frac{\gamma e^{-\delta n}}{(n^2+\beta^2)^{1/2}} dn$$
$$= \frac{\gamma}{2}\left[S_0(\delta\beta) - \pi E_0(\delta\beta) - \pi N_0(\delta\beta)\right]$$

Where $S_\nu(z) =$ Schlafli's polynomial of order ν, $S_0(z) = 0 \forall z)$, (see [11]). $E_\nu(z) =$ Weber's function of order ν, (see [11]) and $N_\nu(z) =$ Neumann's function of order ν, (see [11]). So that

$$I_1(0) = \frac{\gamma}{2}\left[S_0(\delta\beta) - \pi E_0(\delta\beta) - \pi N_0(\delta\beta)\right]$$
$$+ \int_0^\infty \frac{\partial}{\partial\alpha}\left(\frac{\gamma e^{-\delta n}}{((\alpha-n)^2+\beta^2)^{1/2}}\right)|_{\alpha=0} \, dn + O(\alpha^2)$$

now

$$I_0^{'}(0) = \gamma \int_0^\infty \frac{n e^{-\delta n}}{(n^2+\beta^2)^{3/2}} dn$$

$$\Rightarrow I_1(\alpha) = \frac{\gamma}{2}\left[S_0(\delta\beta) - \pi E_0(\delta\beta) - \pi N_0(\delta\beta)\right]$$
$$+ \gamma\alpha \int_0^\infty \frac{n e^{-\delta n}}{(n^2+\beta^2)^{3/2}} dn + O(\alpha^2)$$

Furthermore

$$f_1(0) = \frac{\gamma}{(\alpha^2+\beta^2)^{1/2}} \text{ and } f_1(\infty) = 0$$

$$f_1^{'}(0) = -\gamma.\frac{(\alpha^2\delta - \alpha + \beta^2\delta)}{(\alpha^2+\beta^2)^{3/2}} \text{ and } f_1^{'}(\infty) = 0$$

So that

$$\sum_{n=0}^{\infty} f_1(n) = \frac{\gamma}{2}\left[S_0(\delta\beta) - \pi E_0(\delta\beta) - \pi N_0(\delta\beta)\right]$$

$$+\gamma\alpha\int_0^{\infty}\frac{ne^{-\delta n}}{(n^2+\beta^2)^{3/2}}dn + \frac{1}{2}.\left[\frac{\gamma}{(\alpha^2+\beta^2)^{1/2}}\right]$$

$$+\frac{\gamma}{12}\left[\frac{\alpha^2\delta - \alpha + \beta^2\delta}{(\alpha^2+\beta^2)^{3/2}}\right] + O(\alpha^2).$$

Now considering

$$\sum_{n=0}^{\infty} f_2(n) = \int_0^{\infty} f_2(n)dn + \frac{1}{2}\left[f_2(0) - f_2(\infty)\right]$$

$$\frac{1}{12}\left[f_2^{'}(\infty) - f_2^{'}(0)\right] - \frac{1}{720}[f_2^{'''}(0) - f_2^{'''}(\infty)] + \ldots$$

With similar manipulation as just described it can be shown that

$$\sum_{n=0}^{\infty} f_2(n) = \frac{\gamma}{2}\left[S_0(\delta\beta) - \pi E_0(\delta\beta) - \pi N_0(\delta\beta)\right]$$

$$-\gamma\alpha\int_0^{\infty}\frac{ne^{-\delta n}}{(n^2+\beta^2)^{3/2}}dn + \frac{1}{2}.\left[\frac{\gamma}{(\alpha^2+\beta^2)^{1/2}}\right]$$

$$+\frac{\gamma}{12}\left[\frac{\alpha^2\delta - \alpha + \beta^2\delta}{(\alpha^2+\beta^2)^{3/2}}\right] + O(\alpha^2).$$

So that

$$S = \frac{\gamma}{2}\left[S_0(\delta\beta) - \pi E_0(\delta\beta) - \pi N_0(\delta\beta)\right]$$

$$+\frac{1}{6}.\left[\frac{\gamma\delta}{(\alpha^2+\beta^2)^{1/2}}\right] + O(\alpha^2).$$

To proceed with this method these special functions must be written in a form so that they can be integrated over the volume of interest.

V. Neumann's function, Bessel function of the second kind

Here the Bessel function of the second kind has been obtained, taking the definition of the Neumann function as

$$N_\upsilon(x) = \frac{\cos\upsilon\pi J_\upsilon(x) - J_{-\upsilon}(x)}{\sin\upsilon\pi}$$

Evaluating $N_n(x)$ by l'Hopital's rule for indeterminant forms (i.e. for $\upsilon = n$ (integer) gives

$$N_n(x) = \frac{1}{\pi}\left[\frac{\partial}{\partial\upsilon}J_\upsilon(x) - (-1)^n\frac{\partial}{\partial\upsilon}J_{-\upsilon}(x)\right]\Big|_{\upsilon=n}$$

With

$$J_n(x) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(n+m+1)}.\left(\frac{x}{2}\right)^{2m+n}$$

Using

$$\frac{d}{d\nu}(x^\upsilon) = x^\upsilon\log_e(x)$$

and

$$\frac{d}{dz}(\Gamma(z) = \Gamma\frac{d}{dz}(\log(\Gamma(z)))$$

giving

$$N_n(x) = \frac{2}{\pi}J_n(x)\log_e(\frac{x}{2})$$
$$-\frac{1}{\pi}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!(n+r)!}.\left(\frac{x}{2}\right)^{n+2r}(F(r)+F(n+r))$$
$$-\frac{1}{\pi}\sum_{r=0}^{n-1}\frac{(n-r-1)!}{r!}.\left(\frac{x}{2}\right)^{-n+2r}$$

Where F(r) and F(n+r) are the digamma functions, [1] arising from the differentiation of the gamma function when expressed as an infinite limit. Using properties of the digamma function gives:

$$N_n(x) = \frac{2}{\pi}\left(\log_e(\frac{x}{2}) + \gamma' - \frac{1}{2}\sum_{p=1}^{n}\frac{1}{p}\right)J_n(x)$$
$$-\frac{1}{\pi}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!(n+r)!}.\left(\frac{x}{2}\right)^{n+2r}\sum_{p=1}^{r}\left(\frac{1}{p}+\frac{1}{p+n}\right)$$
$$-\frac{1}{\pi}\sum_{r=0}^{n-1}\frac{(n-r-1)!}{r!}.\left(\frac{x}{2}\right)^{-n+2r}$$

Where γ' is the Euler-Mascheroni constant (as shown in [1]). So finally for n=0 the limiting value is:

$$N_0(x) = \frac{2}{\pi}\left(\log_e(x) + \gamma' - \log_e(2)\right) + O(x^2).$$

VI. The Weber function and its relation to the Struve function.

By definition the Weber function may be expressed as

$$E_\nu(x) = \frac{1}{\pi}\int_0^\pi \sin(\upsilon\vartheta - z\sin\vartheta)d\vartheta$$

The relationship between Weber's function and the Struve function is, for n being a positive integer or zero (see for example [1])

$$E_\nu(x) = \frac{1}{\pi}\sum_{k=0}^{(n-1)/2}\frac{\Gamma(k+\frac{1}{2})\left(\frac{1}{2}z\right)^{n-2k-1}}{\Gamma(n+\frac{1}{2}-k)} - H_n(z)$$

Where $H_n(z)$ is the Struve function defined by

$$H_\nu(x) = (\frac{1}{2})^{\upsilon+1}\sum_{k=0}^{\partial\infty}\frac{(-1)^k z^{2k}}{\Gamma(k+\frac{3}{2})\Gamma(k+\upsilon+\frac{3}{2})}\left(\frac{1}{2}\right)^{2k}$$

It follows that

$$E_0(z) - H_0(z)$$

$$\Rightarrow E_0(z) = -\frac{2}{\pi}(z - \frac{z^3}{1^2 3^2} + \frac{z^5}{1^2 3^2 5^2} - ......)$$

This gives

$$S = 2\gamma\delta\beta - 2\gamma\left[\log_e(\delta\beta) + \gamma' - \log_e(2))\right] + \frac{1}{6}.\left[\frac{\gamma\delta}{(\alpha^2+\beta^2)^{1/2}}\right] + O(\alpha^2).$$

Where to avoid confusion the Euler-Mascheroni constant has been denoted by γ' and $\gamma = x\cos\vartheta$. Thus integration over the volume of interest can now be performed. That is

$$A_\varphi(r,z) = \frac{\mu_0 j}{4\pi}\int_a^b\int_0^{2\pi}\int_\varepsilon^{1-\varepsilon}\{2\gamma\delta\beta - 2\gamma\left[\log_e(\delta\beta) + \gamma' - \log_e(2))\right] + \frac{1}{6}.\left[\frac{\gamma\delta}{(\alpha^2+\beta^2)^{1/2}}\right]\}dxd\vartheta dz' + O(\alpha^2).$$

VII. Considering the Order $\gamma\delta$ term in the expression for $A_\varphi(r,z)$

Considering the $O(\gamma\delta)$ term and denoting this as

$$\Theta = \frac{\mu_0 j\delta}{24\pi}\int_a^b\int_0^{2\pi}\int_\varepsilon^{1-\varepsilon}\frac{\gamma}{(\alpha^2+\beta^2)^{1/2}}dxd\vartheta dz' \quad (10)$$

Performing the ϑ integration first gives

$$\Theta = \frac{\mu_0 j\delta}{24\pi}\int_a^b\int_\varepsilon^{1-\varepsilon}xdxdz'\int_0^{2\pi}\frac{\cos\vartheta}{(\lambda^2 - \eta\cos\vartheta)^{1/2}}d\vartheta$$

Where $\lambda^2 = (z-z')^2 + x^2 + r^2$ and $\eta = 2xr$. Slight manipulation leads to

$$\Theta = \frac{\mu_0 j\delta}{24\pi}\int_a^b\int_\varepsilon^{1-\varepsilon}\frac{x}{\mu}dxdz'\int_0^{\pi/2}\frac{2\sin^2 u - 1}{(1-k^2\sin^2 u)^{1/2}}du$$

where $\mu^2 = \lambda^2 + \eta = (z-z')^2 + (x+r)^2$ and $k^2 = \frac{4xr}{(z-z')^2 + (x+r)^2}$, with $\frac{\vartheta}{2} = \frac{\pi}{2} - u$.

It can be shown that (Gradsteyn and Ryzhik [7])

$$\int_0^{\pi/2}\frac{\sin^{2\mu-1}x\cos^{2\nu-1}x}{(1-k^2\sin^2 x)^\rho}dx = \frac{1}{2}B(\mu,\nu)F(\rho,\mu,\mu+\nu,k^2)$$

Where B(m,n) is the Beta function and $F(a,b,c,z^2)$ is the Hypergeometric function, so that

$$\int_0^{\pi/2}\frac{2\sin^2 u - 1}{(1-k^2\sin^2 u)^{1/2}}du = B(\frac{3}{2},\frac{1}{2})F(\frac{1}{2},\frac{3}{2},2,k^2) - \frac{1}{2}B(\frac{1}{2},\frac{1}{2})F(\frac{1}{2},\frac{1}{2},1,k^2)$$

So that

$$\Theta = \frac{\mu_0 j\delta}{6\pi}B(\frac{3}{2},\frac{1}{2})\int_a^b\int_\varepsilon^{1-\varepsilon}\frac{x}{((z-z')^2+(x+r)^2)^{1/2}}. * F(\frac{1}{2},\frac{3}{2},2,k^2)dxdz'$$

$$- \frac{\mu_0 j\delta}{12\pi}B(\frac{1}{2},\frac{1}{2})\int_a^b\int_\varepsilon^{1-\varepsilon}\frac{x}{((z-z')^2+(x+r)^2)^{1/2}}. * F(\frac{1}{2},\frac{1}{2},1,k^2)dxdz', \quad (11)$$

with

$B(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(n+m)}$, it can also be shown that

$B(\frac{3}{2},\frac{1}{2}) = \frac{\pi}{2}$ and $B(\frac{1}{2},\frac{1}{2}) = \pi$.

In [10] it has been shown that these integrals containing the series of the hypergeometric function are uniformly convergent in the interval of integration so that with some algebraic manipulation it can be shown [10] that

$$\Theta = \frac{\mu_0 j\delta}{12}\int_a^b\int_\varepsilon^{1-\varepsilon}\{\frac{x}{((z-z')^2+(x+r)^2)^{1/2}}. * \sum_{n=0}^{\infty} E_n \frac{k^{2n}}{n!}\} dxdz$$

Where $E_n = C_n - D_n$ and

$$C_n = \frac{(\frac{3}{2},n)(\frac{3}{2},n)}{(2,n)}, D_n = \frac{(\frac{3}{2},n)(\frac{1}{2},n)}{(1,n)} \text{ with}$$

$$(\lambda,k) = \frac{\Gamma(\lambda,k)}{\Gamma(\lambda)} = \lambda(\lambda+1).....(\lambda+k-1), k \geq 0.$$

VIII. Considering the Order k^0 Term in the Expression for Θ.

Considering the term and denoting this integral as K_0 that is:

$$K_0 = \frac{\mu_0 jE_0}{12}\int_a^b\int_\varepsilon^{1-\varepsilon}\frac{x}{((z-z')^2+(x+r)^{1/2}} dxdz'$$

Thus

$$K_0 = -\frac{\mu_0 jE_0}{12}[[(\frac{u^2}{2} - ru)\log_e(\sigma + (\sigma^2+u^2) + \frac{\sigma}{2}(\sigma^2+u^2)^{1/2} - r\sigma\log_e(u+(\sigma^2+u^2)^{1/2}]_{a+r}^{b+r}]_{z-\varepsilon-n}^{z-1+\varepsilon-n}$$

Where u = x + r and $\sigma = z - z'$.

IX. Considering the Order k^2 Term in the Expression for Θ.

Considering the $O(k^2)$ term and denoting this term as K_2, say where:

$$K_2 = \frac{\mu_0 jE_1}{3} r\int_a^b\int_\varepsilon^{1-\varepsilon}\frac{x^3}{((z-z')^2+(x+r)^{3/2}} dxdz'$$

Computing these integrals gives

$$K_2 = -\frac{\mu_0 jE_1}{3} r[[w(w^2+u^2)^{1/2} + 3ru\log_e(w+(w^2+u^2)^{1/2}) - 3rw\log_e(w+(w^2+u^2)^{1/2}) + 3r(w^2+u^2)^{1/2} - 3r^2\log_e(w+(w^2+u^2)^{1/2}) + \frac{r^3(w^2+u^2)^{1/2}}{uw})]_{a+r}^{b+r}]_{z-\varepsilon}^{z-1+\varepsilon}$$

Where u=x+r and w=z-z'. Therefore

$$A_\varphi(r,z) = \frac{\mu_0 j}{4\pi}\int_a^b\int_0^{2\pi}\int_\varepsilon^{1-\varepsilon}\{2\gamma\delta\beta - 2\gamma[\log_e(\delta\beta)+\gamma' - \log_e(2)]\} dxd\vartheta dz' + K_0 + K_2 + O(\alpha^2). \quad (12)$$

X. Considering the Order $\delta\beta^0$ Term in the Expression for $A_\varphi(r,z)$.

Considering the $O(\delta\beta^0)$ term in equation (12) and denoting this term by Δ_0, say where

$$\Delta_0 = -\frac{\mu_0 j}{2\pi}(\gamma' - \log_e(2))\int_a^b\int_0^{2\pi}\int_\varepsilon^{1-\varepsilon} x\cos\vartheta dxd\vartheta dz' = 0$$

So that

$$A_\varphi(r,z) = \frac{\mu_0 j}{4\pi}\int_a^b\int_0^{2\pi}\int_\varepsilon^{1-\varepsilon}\{2\gamma\delta\beta - 2\gamma\ln\delta\beta\} dxd\vartheta dz' + K_0 + K_2 + O(\alpha^2).$$

XI. Considering the Order $\delta\beta$ and γ Terms in the Expression for $A_\varphi(r,z)$.

Considering the $O(\delta\beta)$ and $O(\gamma)$ terms and denoting this integral as

$$\Delta_1 = \frac{\mu_0 j}{2\pi}(1-2\varepsilon)\int_a^b \int_0^{2\pi} (\delta x \cos\vartheta (x^2 + r^2 - 2xr\cos\vartheta)^{1/2} - \Gamma x \cos\vartheta) dx d\vartheta$$

Where $\Gamma = \log_e |\delta\beta|$. With slight manipulation it can be shown that

$$\Delta_1 = 4\frac{\mu_0 j}{\pi}(1-2\varepsilon)\delta\int_a^b x(x+r)dx \int_0^{\pi/2} \sin^2 u (1-\lambda^2 \sin^2 u)^{1/2} du$$

$$-2\frac{\mu_0 j}{\pi}(1-2\varepsilon)\delta\int_a^b x(x+r)dx \int_0^{\pi/2} (1-\lambda^2 \sin^2 u)^{1/2} du$$

Where

$$\lambda^2 = \frac{2k^2}{1+k^2}, k^2 = \frac{\eta}{\mu^2}, \mu^2 = x^2 + r^2, \eta = 2xr,$$

$$\frac{\vartheta}{2} = \frac{\pi}{2} - u$$

It can be shown that (see [7])

$$\int_0^{\pi/2} \sin^m u \cos^n u (1-k^2 \sin^2 u)^{1/2} du = \frac{1}{2}B(\frac{m+1}{2}, \frac{n+1}{2})F(\frac{m+1}{2}, -\frac{1}{2}, \frac{m+n+2}{2}, k^2)$$

For $m > -1, n > -1, |k^2| < 1$, where $B(p,q)$ is the Beta function and $F(a,b,c,z^2)$ is the hypergeometric function whose convergence has already been discussed, thus Δ_1 can easily be evaluated. Now the term containing the logarithm of β must be considered, denoting this integral as Δ_2 then

$$\Delta_2 = -\frac{\mu_0 j}{4\pi}(1-2\varepsilon)\int_a^b x dx \int_0^{2\pi} \cos\vartheta (x^2 + r^2 - 2xr\cos\vartheta) d\vartheta$$

Once again this integral has be computed see Pavlika [10], thus finally

$$A_\varphi(r,z) = K_0 + K_1 + \Delta_1 + \Delta_2 + O(\alpha^2)$$

Where K_0, K_2, Δ_1 and Δ_2 are now known.

XII. Conclusions.

The two methods of solution were found to be in good agreement however more terms are required for the method of solution based on the Euler-Maclaurin summation formula. The effect of the permeability of the iron is shown in figures 4, 5, 6 and 7.

XIII. Tables

Table II
Values Of Aϕ(R,Z) Using The Using The Quaternion Method Of Solution

r	Z	μ=10^3	μ=10^2	μ=10	μ=1
0	0.1	0	0	0	0
0.1	0.1	0.89172	0.881238	0.7576	0.3481
0.2	0.1	1.79492	1.762867	1.5141	0.6902
0.3	0.1	2.69390	2.645277	2.2679	1.0201
0.4	0.1	3.59466	3.528858	3.0178	1.3319
0.5	0.1	4.49780	4.414002	3.7625	1.6196
0.1	0.2	0.89782	0.882508	0.7642	0.3733
0.1	0.3	0.89596	0.883737	0.7693	0.3926
0.1	0.4	0.89920	0.884629	0.7726	0.4049
0.1	0.5	0.89943	0.884955	0.7738	0.4091

Table III
Values of Br(r,z) using the Quaternion Method of Solution

r	Z	μ=10^3	μ=10^2	μ=10	μ=1
0.1	0.1	5.832E-3	0.0163	0.1042	0.0362
0.2	0.1	1.315E-2	0.0343	0.2120	0.0776
0.3	0.1	2.344E-2	0.0556	0.3674	0.1426
0.4	0.1	3.819E-2	0.0820	0.4521	0.1599
0.5	0.1	5.887E-2	0.1151	0.5914	2.0972
0.1	0.2	8.426E-3	0.0166	0.0852	0.2937
0.1	0.3	8.083E-3	0.0136	0.0607	0.2072
0.1	0.4	4.898E-3	0.0071	0.0316	0.0107
0.1	0.5	0	0	0	0

Table IV
Values of Bz(r,z) using using the Quaternion Method of Solution

r	Z	μ=10^3	μ=10^2	μ=1
0	0.1	17.9170	17.6164	6.9822
0.1	0.1	17.0150	17.6151	7.0023
0.2	0.1	17.9091	17.6112	7.0628
0.3	0.1	17.8991	17.6047	7.1635
0.4	0.1	17.8852	17.5965	7.3046
0.5	0.1	17.8673	17.5839	7.4860
0.1	0.2	17.9732	17.6546	7.5233
0.1	0.3	17.9723	17.6771	7.9259
0.1	0.4	17.9861	17.6996	8.1803
0.1	0.5	17.9867	17.7015	8.2673

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Dr Vasos Pavlika has a B.Sc (Hons) in Mathematics and Physics (1987), an M.Sc in Mathematics (1988), a Ph.D in Applied Mathematics (1995) and a PGCE in teaching Mathematics and Physics in Post-Compulsory Education (1999). He has worked at the University of North London (1988), Newham Sixth Form College (1999), Havering College of Further and Higher Education (2000), The University of East London (2000), The London School of Economics and Political Science (2002-present), The University of Gloucestershire (2002) and The University of Westminster (2003-2010), SOAS, University of London (2003-Present), The University of Oxford (2004-Present), St George's, University of London (2011-2013). Dr Pavlika is a member of the WSEAS, an Editor of the Mathematical Journal of the Antarctica and a member of the WCNA. He has a certificate of Merit for a paper presented at the International MultiConference of Engineers and Computer Scientists 2007, Hong-Kong 21-23 March 2007 and Winner of the Best Paper Award for a paper presented at the International MultiConference of Engineers and Computer Scientists 2007, San Francisco 24-26 October 2007 and has over 20 publications including chapters in the following books: The Calculation of Annular Duct Geometries by Prescribing a Velocity Distribution on the Pressure Surface and the Radius of the Suction Surface", pp2375-2383, *Lecture Notes in Engineering and Computer Science*, Newswood Limited 2007, ISBN 978-988-98671-4-0, Magnetostatic field calculations associated with thick Solenoids in the Presence of Iron using a Power Series expansion and the Complete Elliptic Integrals" *Applied Numerical Methods*, Springer Verlag 2007 and The calculation of axisymmetric duct geometries for incompressible rotational flow with blockage effects and body forces". *Advances in Computational Algorithms and Data Analysis*, Springer Verlag 2008. His research Interests include: teaching in HE, theoretical and computational fluid dynamics and Magnetostatics. He is a Fellow of the Institute of Mathematics and its Applications (FIMA, CMath, CSci and CMathTeach), a Fellow of the Institution of Analysts and Programmers (FIAP), a Member of the Institute of Physics (MInstP, CPhys), a Fellow of the British Computer Society (FBCS, CITP and CEng), a Member of the Institution of Electrical Engineers (MIEE and MIET) and a Member of the British Society for the History of Mathematics.

XIV. Figures

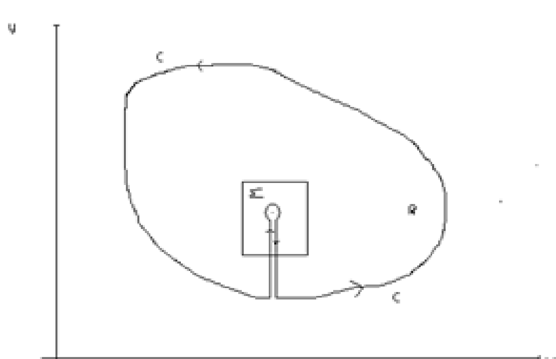

Fig. 1 The region R bounded by the curve C showing the singularity z_0 inside R

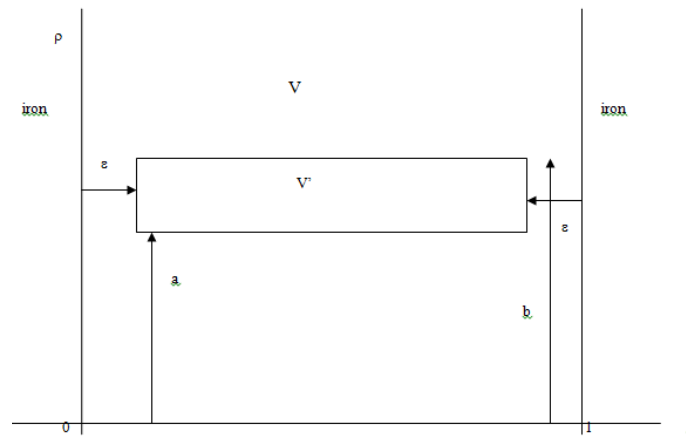

Fig. 2 A toroidal conductor V' of rectangular cross section located midway between two semi infinite regions of iron of finite permeability. The region V is assumed to be insulating.

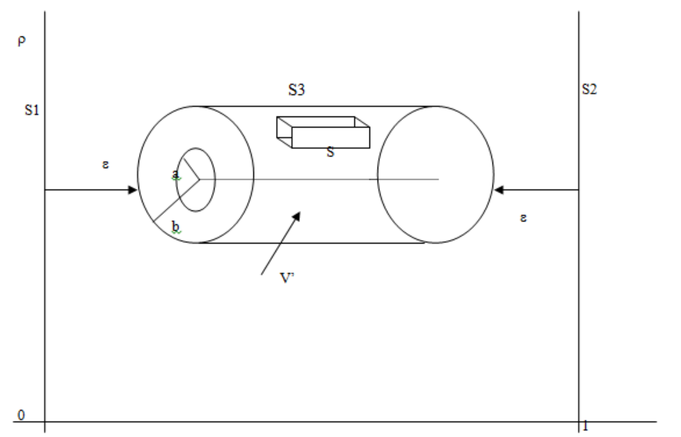

Fig. 3 The volume of interest over which the integrations are performed

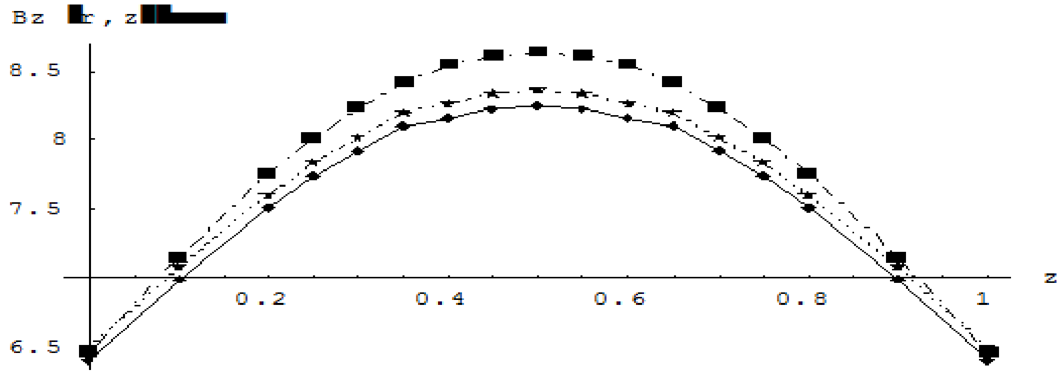

Fig. 4 The variation of $B_z(\rho,z)$ with ρ and z for two semi-infinite regions of iron of unit permeability. +:ρ=0.3, ™:ρ=0.2, •:ρ=0.1 •:ρ=0.3

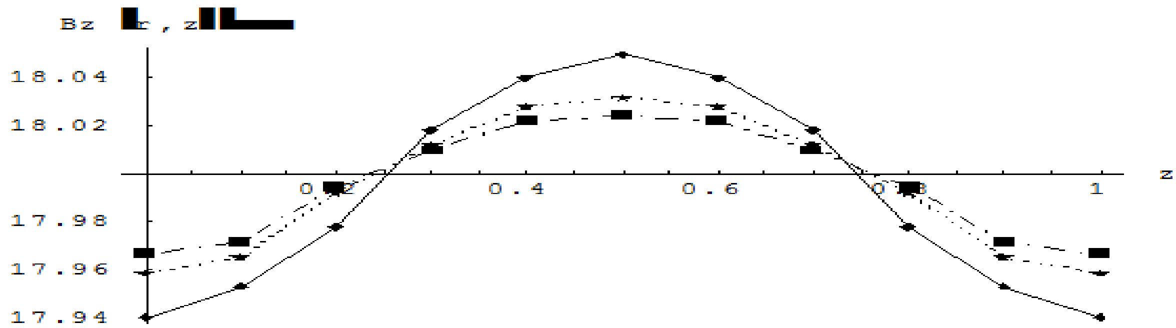

Fig. 5 The variation of $B_z(\rho,z)$ with ρ and z for two semi-infinite regions of iron of infinite permeability. +:ρ=0.1, ™:ρ=0.2,

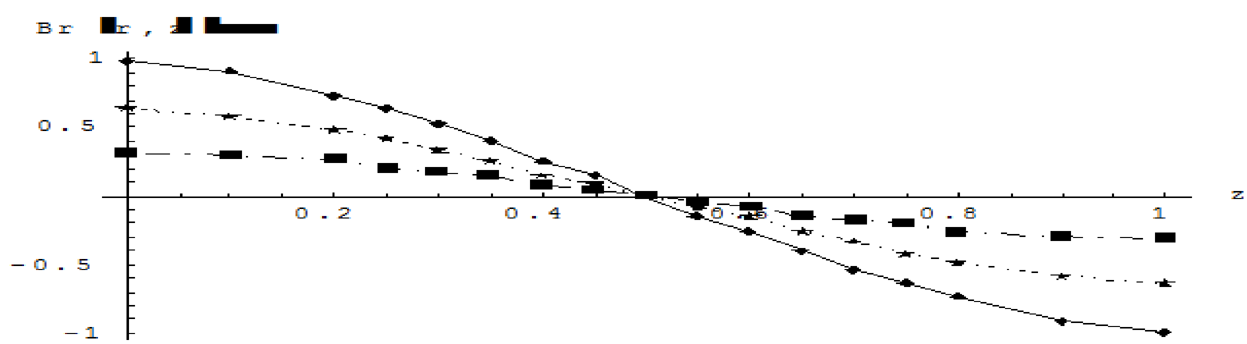

Fig. 6 The variation of $B_r(\rho,z)$ with ρ and z for two semi-infinite regions of iron of unit permeability. +:ρ=0.1, ™:ρ=0.2, •:ρ=0.3

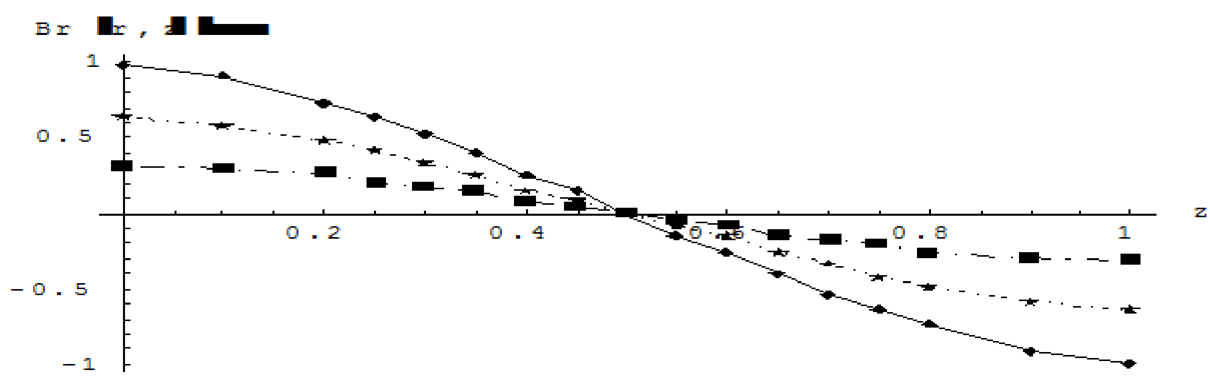

Fig. 7 The variation of $B_\rho(\rho,z)$ with ρ and z for two semi-infinite regions of iron of infinite permeability. +:ρ=0.1, ™:ρ=0.2, •:ρ=0.3